MAY HUMAN CAPITAL RESCUE THE EMPTY PLANET?

Spyridon Boikos
University of Macedonia (Department of Economics),
Thessaloniki, Greece.

Alberto Bucci
Department of Economics, Management and Quantitative
Methods (DEMM) - University of Milan, and ICEA (International
Center for Economic Analysis, Canada

Tiago Miguel Guterres Neves Sequeira
University of Coimbra, Centre for Business and Economics
Research, CeBER and Faculty of Economics

CeBER Working Papers
No. 9 / 2022

CeBER is funded by the Foundation for Science and Technology, I.P.

MAY HUMAN CAPITAL RESCUE THE *EMPTY PLANET*? *

Spyridon Boikos† Alberto Bucci‡

Tiago Neves Sequeira§

Abstract

Recent evidence suggests that fertility rates are (and will be expected to remain in the future) below the replacement level for several countries and especially for the most technological advanced ones, which indicates that the World population will start decreasing sooner or later. In the light of this, we reconsider the Empty Planet result – Jones (2022) – and include human capital and class size effects in R&D endogenous growth models with decreasing population. We find that the introduction of human capital mitigates, or even overcomes, the Empty Planet result. In particular, under some mild conditions, our setting allows obtaining simultaneous long-run economic growth and secular productivity stagnation.

JEL Classification: O30, O40, E13, E17, E61.

Key Words: Endogenous economic growth, R&D, Human Capital, Declining Population, Empty Planet.

*This work has been funded by national funds through *FCT – Fundacao para a Cienciae a Technologia, I.P., Project UIDB/05037/2020.*

†University of Macedonia (Department of Economics), Thessaloniki, Greece. E-mail: **boikos@uom.edu.gr**

‡ ICEA (International Centre for Economic Analysis), Canada, and University of Milan (Department of Economics, Management and Quantitative Methods, DEMM), Italy. E-mail: **alberto.bucci@unimi.it**

§University of Coimbra (CeBER, Faculty of Economics), Coimbra, Portugal. E-mail: **tiago.n.sequeira@fe.uc.pt**

1. Introduction

Some evidence, as well as recent literature, highlights the fact that current fertility rates tend to a level that is definitely below the replacement level, indicating that in the long-run the plausible scenario would be that of a population decrease.

In a very recent paper, Jones (2022) analyzed the consequences of a declining population for economic growth within the endogenous-growth setting with scale-effects (à la Romer, 1990) as well as the semi-endogenous one (Jones, 1995). The predictions of these models point to stagnation of growth in the very long-run (the empty planet result). For example, one of these predictions shows that the long-run level of GDP per person would be around 60-90% higher than current income and then stagnates at that level. The intuition of the result is simple. The model uses labor to be allocated to the R&D sector, the ultimate engine of growth. As increasing population fuels growth, so decreasing population would lead to stagnation. Then, the author characterizes the decentralized and social planner equilibria of his model with endogenous fertility. Besides showing that declining population may be the optimal outcome of people's choices, it is also showed that the planner problem can involve multiple steady states. [1] Despite the interest of the

[1] Sequeira et al. (2018), using a stylized Schumpeterian endogenous growth model, obtain either fully endogenous growth without scale effects or stagnation. The possibility of stagnation in their paper is due to the increase in complexity.

analysis, results depend on the crucial assumption that the knowledge production function includes only raw labor (directly depending on population) which is not a reproducible factor (e.g., Rebelo, 1991). One of the major advances in the endogenous growth theory after the scale-free breakthrough lead by Jones (1995) and Dinopoulos and Thompson (1999, 2000) was the inclusion of human capital as an additional engine of growth that interacts together with R&D (Arnold, 1998, Funke and Strulik, 2000). Human capital enters as an input in the knowledge production function, substituting the role of raw labor. This implies that the economy has a reproducible factor of production influencing R&D.

Our main aim in this paper is to study the long-run growth predictions of endogenous growth models with R&D and human capital accumulation in the presence of negative population growth. To this end, we set-up endogenous growth models with both R&D and human capital accumulation and negative population growth and study their long-run properties. An additional feature we introduce to the models is the possibility of positive class-size effects induced by a declining population on the accumulation of human capital (see e.g. Mathis, 2017; Bucci, 2022).

Our findings indicate that, despite these models do not imply the possibility of stagnation in real GDP *per capita*, those including a class size effect allow for the possibility of stagnation in the level of technology. Moreover, when the models are taken to data with several parameterizations, we obtain that the level of technology should stagnate in the very long-run. This conclusion is not negligible for three main reasons, and is different from the results and the focus of the main related papers (namely, Jones, 2022; Bucci, 2022).

First, a stream of very recent literature suggests that TFP tends to long-run stagnation. For instance, in Kasparov et al. (2012), the authors point out that the collapse of advanced country growth is not merely a result of the financial crisis. They argue that these countries' weakness reflects secular stagnation in technology and innovation. Fernald (2014) presents evidence according to which labor and total factor productivity (TFP) growth slowed prior to the great recession. He argues that it marked a retreat from the exceptional, but temporary, information technology-fueled pace from the mid-1990s to early in the twenty-first century. Gordon (2012) also argues that the slowdown happening after the 2000s roots at the features of the innovations (computers, the web, mobile phones) of the late XXth century, which spillovers were less long-lasting than the ones of the innovations of the second industrial revolution and anticipates scenarios of long anemic economic growth or even stagnation (see also Bloom et al., 2017). More recently, Brinca et al. (2017) pointed to the efficiency wedge (measured by A in the final-good production function) as the main source of the great recession at the end of the 2000s decade.

Second, we provide an additional mechanism (the class-size effect that can be quite reasonable) that contributes to that possibility. Last but not least, TFP stagnation in the long-run does not imply GDP per capita stagnation in the long-run as previous literature, mentioned above, tends to indicate.

In a future work, companion to the present one, we plan to study the social planner equilibrium with endogenous fertility as well as to quantify the class-size externality *vis-à-vis* the usual externalities coming from the production of knowledge: the standing-on-shoulders effect and the stepping-on-the toes effect.

In the next section we present the models. We divide the model section in three subsections. First, we present the model without the possibility of class-size effects and study their long run properties. Second, we present the model with the possibility of class-size effects and study their long run properties. Third, we take the model to data through calibration and present quantitative results both for the long-run and for transitional dynamics. In the third Section, we conclude. We also provide an Appendix with all the detailed proofs.

2. The Basic Setup: Exogenous Human Capital Accumulation and Negative Population Growth

We embed exogenous human capital accumulation into an integrated R&D-based growth model able to reconcile the fully-endogenous-growth approaches by Romer (1990) and Grossman and Helpman (1991) with the Jones (1995 and 2022)'s semi-endogenous-growth version of these models. At this aim, we postulate that the production of goods $\left(Y_t\right)$ and the invention of new ideas $\left(A_t\right)$ are obtained through reproducible human capital $\left(H_t\right)$, such that:

$$Y_t = A_t^{\sigma} H_t, \qquad \sigma > 0. \tag{1}$$

In Eq. (1), Y_t is aggregate real GDP (the *numeraire*), H_t is aggregate human capital, and A_t is the number of existing ideas at *t*. In per capita terms, Eq. (1) can be recast as: $y_t = A_t^{\sigma} h_t$, with $h_t = \frac{H_t}{N_t}$ and $y_t = \frac{Y_t}{N_t}$ representing per capita human capital and per capita GDP, respectively, and N_t being the size of population at *t*. We assume that population grows over time at an exogenous rate $\left(\dot{N}_t / N_t\right)$ and that such a rate is negative, *i.e.*:

$$\frac{\dot{N}_t}{N_t} = -n < 0 \qquad \text{or} \qquad N_t = N_0 e^{-nt}, \qquad n > 0.$$[2]

Per capita human capital evolves over time according to:

$$\dot{h}_t = \xi h_t - \left(\delta - n\right) h_t \tag{2}$$

where $\xi > 0$ is the efficiency of human capital accumulation, $\delta \in \left(0,1\right)$ is the depreciation rate of human capital, and n is the negative of the population growth rate, *i.e.* $n = -\frac{\dot{N}_t}{N_t}$. At this stage, to keep things as much simple as possible, we assume that there is no allocation of human capital between

[2] As in Jones (2022), our model features an exogenous negative growth rate of population. In an Appendix we show that a model with endogenous fertility and endogenous choice of human capital can indeed yield a negative growth rate of population, as well.

production and educational activities. In Eq. (2), a negative growth rate of population ($n = -\dot{N}_t / N_t$) now increases per capita human capital growth, as the term $n \cdot h_t$ turns out to be positive. We shall come back to the economic rationale of this observation in Section 2.1, where we formally account for the possible existence of *class-size effects* in the process of students' skill-acquisition.

As in the canonical semi-endogenous growth literature (Jones, 1995 and 2022), the time-evolution of the stock of ideas (A_t) is described by the following differential equation:

$$\dot{A}_t = \alpha A_t^{1-\beta} H_t^{\lambda}, \qquad \alpha > 0, \qquad 0 < \lambda \leq 1, \qquad 1-\beta \leq 1 \tag{3}$$

where λ and $1-\beta$ measure the extent of the duplication and intertemporal spillover effects of research, respectively. Eq. (3) can be immediately recast as $\dot{A}_t = \alpha A_t^{1-\beta} \left(N_t h_t\right)^{\lambda}$, which best emphasizes the fact that in our model-economy (unlike Jones, 2022) human capital per capita can be purposefully accumulated over time through formal education (Eq. 2). If we set $\beta = 0$ and $\lambda = 1$, our model is also able to embed the fully endogenous growth literature displaying strong scale effects in population size (*e.g.*, Romer, 1990; Aghion and Howitt, 1992; Grossman and Helpman, 1991), as in this case the growth rate of ideas would be written as: $\frac{\dot{A}_t}{A_t} = \alpha h_t N_t$.

Following the same steps in Jones (2022, p. 41), by integrating the differential equation:

$$\int A_t^{\beta-1} dA_t = \int \alpha \left[\underbrace{\left(N_0 e^{-nt}\right)}_{N_t} \cdot \underbrace{\left(h_0 e^{(\xi-\delta+n)t}\right)}_{h_t} \right]^{\lambda} dt,$$

it is immediately obvious that we have now two different effects of the negative growth rate of population, n: (i) On the one hand, it decreases the total amount of resources (population size, N_t) that can potentially be allocated to innovation, so contributing to the empty planet (stagnation) result, whereas (ii) On the other hand, it increases the quality of existing people (h_t), which has an opposing effect on the empty planet (stagnation) result. As far as the rate of innovation is concerned, in this simple model these two effects cancel each other out. The following proposition presents the main result both for the semi-endogenous growth model (with exogenous human capital accumulation and without scale effects of population), and the fully-endogenous growth model (with exogenous human capital accumulation and potential scale effects of population).

Proposition 1

(a) In the semi-endogenous growth case – $\beta > 0$ *and* $0 < \lambda < 1$ *in Eq.* (3) *– the differences in the stock of knowledge* $\frac{A_t}{A_0}$ *and in the output per capita* $\frac{y_t}{y_0}$ *are given, respectively, by:*

$$\frac{A_t}{A_0} = \left\{ 1 - \frac{\beta g_{A0}}{\lambda(\xi-\delta)} \left[1 - e^{\lambda(\xi-\delta)t} \right] \right\}^{1/\beta} \tag{4}$$

$$\frac{y_t}{y_0} = \left\{ 1 - \frac{\beta}{\sigma\lambda} \left(\frac{g_{y0} - n}{\xi - \delta} - 1 \right) \left[1 - e^{\lambda(\xi-\delta)t} \right] \right\}^{\sigma/\beta} \cdot e^{(\xi-\delta+n)t}, \qquad g_{y0} = \sigma g_{A0} + \xi - \delta + n \tag{5}$$

(b) In the fully endogenous growth case – $\beta = 0$ *and* $\lambda = 1$ *in Eq.* (3) *– the differences in the stock of knowledge* $\frac{A_t}{A_0}$ *and in the output per capita* $\frac{y_t}{y_0}$ *are given, respectively, by:*

$$\frac{A_t}{A_0} = e^{\frac{g_{A0}}{(\xi-\delta)}\left[e^{(\xi-\delta)t}-1\right]} \tag{6}$$

$$\frac{y_t}{y_0} = \left\{ e^{\frac{1}{\sigma}\left(\frac{g_{y0}-n}{\xi-\delta}-1\right)\left[e^{(\xi-\delta)t}-1\right]} \right\}^{\sigma} \cdot e^{(\xi-\delta+n)t}, \qquad g_{y0} = \sigma g_{A0} + \xi - \delta + n \tag{7}$$

Proof.: See *Appendix A*.

Regarding the semi-endogenous growth model with exogenous human capital accumulation [Part (a) of Proposition 1], Eq. (4) clearly shows that, for any positive initial growth rate of technology, $g_{A0} > 0$, and any positive growth rate of aggregate human capital, $\xi > \delta$, we can observe perpetual growth of A_t. This result stresses the fundamental role played by human capital accumulation in the model. Similarly, in Eq. (5), with $g_{y0} = \sigma g_{A0} + (\xi - \delta) + n$, output per capita always grows whenever $g_{A0} > 0$ and positive aggregate human capital growth is allowed ($\xi > \delta$). At this stage it is worth pointing it out that the presence of a negative growth rate of population $\left(\dot{N}_t / N_t = -n,\ \ n > 0\right)$ contributes to having positive growth of human capital *per capita* even when *total* human capital is not growing, *i.e.* $\xi = \delta$. If we check the behavior of Eqs. (4)–(5) in the very long run (*i.e.*, when $t \to +\infty$), then we can state the following conclusions:

- If $g_{A0} > 0$ and $\xi \geq \delta$,[3] both ratios tend to infinity, $\frac{A_t}{A_0} = \frac{y_t}{y_0} \to +\infty$, and there is perpetual growth in the level of technology and in the level of income per capita together;
- Instead, if $\xi < \delta$, then $\frac{A_t}{A_0} = \left[1 - \frac{\beta g_{A0}}{\lambda\left(\xi - \delta\right)}\right]^{1/\beta} > 0$ and constant, which implies stagnation in the level of technology, but not in the level of income per capita, as long as $\frac{\dot{h}_t}{h_t} = \xi - \delta + n > 0$, implying $\frac{y_t}{y_0} \to +\infty$;
- Finally, if $\xi < \delta$ and $\frac{\dot{h}_t}{h_t} = \xi - \delta + n < 0$, then $\frac{y_t}{y_0} \to 0$, which implies a constant reduction of per capita income towards stagnation, as well.

Overall, the results stated above suggest that in a semi-endogenous growth setting *with* exogenous human capital accumulation we can observe perpetual growth, simultaneously in the level of technology and in the level of real income per capita, if $\xi \geq \delta$, for any $g_{A0} > 0$. However, even if the level of technology stagnates ($\xi < \delta$), we can still observe an unbounded increase in per capita income provided that per capita human capital continues to rise over time, $\frac{\dot{h}_t}{h_t} = \xi - \delta + n > 0$. Yet, at this stage, we must notice that the case for a negative growth rate of total human capital, implied by $\xi < \delta$, is at odds with the available data on human capital accumulation worldwide (see, for

[3] The specific case $\xi = \delta$ is analyzed separately in *Appendix B*.

example, Bils and Klenow, 2000; Feenstra *et al.*, 2015; World Bank, 2021), hence we can safely neglect such a scenario in the remainder of the paper.

As far as the fully-endogenous growth model *with* exogenous human capital accumulation is concerned [Part (b) of Proposition 1], Eq. (6) shows that for a positive initial growth of technology, $g_{A0} > 0$, and a positive growth of total human capital, $\xi > \delta$, we can observe perpetual growth of A_t. This result underscores once again the positive role played in the model by human capital investment. Similarly, in Eq. (7) note that, for any $g_{A0} \geq 0$ and $\xi > \delta$, output per capita always grows because of the growing human capital, and (eventually) of the growing technology, as well. Therefore, contrary to Jones (2022), the inclusion of positive human capital growth at the aggregate level (and, hence, at the per-capita level, too!) allows the economy to escape the pessimistic *Empty Planet Result* both in semi-endogenous and fully endogenous R&D-based growth models.

If we check the behavior of Eqs. (6)–(7) in the very long run (*i.e.*, when $t \to +\infty$), then we can state the following conclusion:

- If $g_{A0} > 0$ and $\xi \geq \delta$,[4] then both ratios tend to infinity, $\frac{A_t}{A_0} = \frac{y_t}{y_0} \to +\infty$, and there is perpetual growth simultaneously in the level of technology and in the level of real income per capita.

In what follows, we extend the basic growth framework analyzed till now in several directions.

[4] The specific case $\xi = \delta$ is analyzed separately in *Appendix B*.

2.1 Accounting for the existence of *class-size effects* in the accumulation of human capital *per capita*

In this section, we analyze the first extension of the basic model described above. In the extension proposed here we take explicitly into account the existence of possible *'class-size-effects'* in the law of motion of human capital per capita. Indeed, research in the field of the economics of education seems now to indicate that there is a positive effect of a smaller class size on an individual student achievement (see, among many others, Bandiera et al., 2010; Konstantopoulos and Chung, 2009; and the recent survey by Mathis 2017). Within the huge class-size debate, the available evidence now shares the view that the effect of a class-size reduction (especially in the first years of schooling) is different across distinct groups of children, and is generally larger for those children coming from minorities or other specific disadvantaged communities. In Hatties (2005, p. 396)'s words the whole class-size literature can be condensed as follows:

> *"…Across these meta-analyses, summaries of major initiatives, and newer studies, the average effect-size is 0.13. Thus, the typical effect of reducing class sizes from 25 to 15 is about 0.10–0.20. Perhaps as interesting as the typical value, is that there is not a lot of variance in these estimates; the mean is a reasonable summary of the effects of reducing class size. These studies represent a variety of designs including meta-analysis, longitudinal studies, cross-cohort studies, are from many countries (USA, UK, Israel, Bolivia), from across all grades, and use some of the most sophisticated statistical methods available. There is remarkable consistency across the effect-sizes from these many diverse studies…"*.

Economically, the existence of a class size effect reinforces the idea that a declining population may have a positive impact on the accumulation of human capital per capita, as a smaller population size (a proxy for a reduced class size) leads to a better student's school-performance. In the model of this section the magnitude of the class size effect is measured by a parameter $0 < \epsilon < 1$ which, by multiplying the negative growth rate of population, contributes to increase the whole productivity of human capital in the production of new human capital. In other words, we now rewrite Eq. (2) as:

$$\dot{h}_t = \left(\xi + \epsilon n\right) h_t - \delta h_t , \qquad 0 < \epsilon < 1. \tag{8}$$

Unlike Eq. 2 (where $\epsilon = 1$), we consider more realistic the case where ϵ is closer to 0 than to 1, so taking explicitly into account the evidence that, on the whole, the effect (on individual schooling performance) of reducing class sizes is generally positive but, at the same time, also rather small. Formally, and for the sake of simplicity, in Eq. (8) the class size effect is introduced as a positive and linear $\left(0 < \epsilon < 1\right)$ effect of the negative growth rate of population $\left(n = -\dot{N}_t / N_t\right)$ on the productivity $\left(\xi\right)$ of the existing human capital $\left(h_t\right)$ in the acquisition of new skills at an individual level $\left(\dot{h}_t\right)$.[5] Using Eq. (8), and following the same steps of the previous section, we come to the following results (Proposition 2).

[5] Class size effects are often included in linear regressions for schooling achievements (*e.g.*, in test scores), as in Bandiera et al. (2010) for example.

Proposition 2 *(accounting for class-size effects)*

(a) In the semi-endogenous growth case – $\beta > 0$ *and* $0 < \lambda < 1$ *in Eq.* (3) *– the differences in the stock of knowledge* $\frac{A_t}{A_0}$ *and in the per person output* $\frac{y_t}{y_0}$ *are given respectively by:*

$$\frac{A_t}{A_0} = \left\{1 - \frac{\beta g_{A0}}{\lambda\left[\xi + (\epsilon - 1)n - \delta\right]} \cdot \left[1 - e^{\lambda\left[\xi + (\epsilon-1)n - \delta\right]t}\right]\right\}^{1/\beta} \tag{9}$$

$$\frac{y_t}{y_0} = \left\{1 - \frac{\beta}{\sigma\lambda}\left[\frac{g_{y0} - n}{\xi + (\epsilon - 1)n - \delta} - 1\right]\left[1 - e^{\lambda\left[\xi + (\epsilon-1)n - \delta\right]t}\right]\right\}^{\sigma/\beta} \cdot e^{(\xi + \epsilon n - \delta)t}, \; g_{y0} = \sigma g_{A0} + \xi - \delta + \epsilon n \tag{10}$$

(b) In the fully endogenous growth case – $\beta = 0$ *and* $\lambda = 1$ *in Eq.* (3) *– the differences in the stock of knowledge* $\frac{A_t}{A_0}$ *and in the per person output* $\frac{y_t}{y_0}$ *are given, respectively, by:*

$$\frac{A_t}{A_0} = e^{\frac{g_{A0}}{\xi + (\epsilon-1)n - \delta}\left[e^{\left[\xi + (\epsilon-1)n - \delta\right]t} - 1\right]} \tag{11}$$

$$\frac{y_t}{y_0} = \left\{e^{\frac{1}{\sigma}\left[\frac{g_{y0} - n}{\xi + (\epsilon-1)n - \delta} - 1\right]\cdot\left[e^{\left[\xi + (\epsilon-1)n - \delta\right]t} - 1\right]}\right\}^{\sigma} \cdot e^{(\xi + \epsilon n - \delta)t}, \qquad g_{y0} = \sigma g_{A0} + \xi - \delta + \epsilon n \tag{12}$$

Proof.: See *Appendix A*.

Thus, in the presence of class-size effects, the stagnation result for the level of technology can occur if and only if $\left[\xi + (\epsilon - 1)n - \delta\right] < 0 \;\Leftrightarrow\; n > \underline{n} = \frac{\xi - \delta}{1 - \epsilon}$, where $\underline{n}$ represents the lower bound of the rate of population decline above which the level of technology will stagnate sometime in the future. The larger the exogenous efficiency of human capital accumulation (ξ) and the smaller the depreciation rate of human capital (δ), the faster the population decline should be to start observing stagnation in technology. On the other hand, the smaller the class size effect $\left(0 < \epsilon < 1\right)$ and the smaller the threshold level of population decline above which technology stagnation will be obtained.

We now focus our attention on the case $g_{A0} > 0$ and $\xi > \delta$,[6] and check the behavior of Eqs. 9–12 in the very long run $\left(i.e., \text{ when } t \to +\infty\right)$.

If $g_{A0} > 0$, $\xi > \delta$,[7] and $\xi + \left(\epsilon - 1\right)n - \delta > 0$,[8] both ratios tend to infinity, $\frac{A_t}{A_0} = \frac{y_t}{y_0} \to +\infty$, and there is perpetual growth in the semi-endogenous as well as in the fully endogenous model.

On the other hand, if $g_{A0} > 0$, $\xi > \delta$, and $\xi + \left(\epsilon - 1\right)n - \delta < 0$, then:

$$\frac{A_t}{A_0} = \left\{1 - \frac{\beta g_{A0}}{\lambda\left[\xi + \left(\epsilon - 1\right)n - \delta\right]}\right\}^{1/\beta} > 0 \ ; \ \frac{y_t}{y_0} = \left\{1 - \frac{\beta}{\sigma\lambda}\left[\frac{\sigma g_{A0}}{\xi + \left(\epsilon - 1\right)n - \delta}\right]\right\}^{\sigma/\beta} \cdot e^{\left(\xi + \epsilon n - \delta\right)t}$$

in the semi-endogenous growth model,

and:

$$\frac{A_t}{A_0} = e^{\frac{g_{A0}}{-\left[\xi + \left(\epsilon - 1\right)n - \delta\right]}} > 0 \qquad ; \qquad \frac{y_t}{y_0} = \left\{e^{\frac{1}{\sigma}\left[-\frac{\sigma g_{A0}}{\xi + \left(\epsilon - 1\right)n - \delta}\right]}\right\}^{\sigma} \cdot e^{\left(\xi + \epsilon n - \delta\right)t}$$

in the fully endogenous growth model.

---

[6] This is done to be consistent with the model without class-size effects (see Proposition 1). Indeed, in that framework, if $g_{A0} > 0$ and $\xi > \delta$, there is growing total (and per capita) human capital, and stagnation (both in the level of technology and in the level of income per capita) can never occur, either in the semi-endogenous or in the fully endogenous growth model. For the sake of completeness, though, *Appendix B* studies the specific case $\xi = \delta$ in the presence of class-size effects in human capital investment, too.

[7] Which implies positive growth of human capital per capita.

[8] Which implies positive growth of aggregate human capital.

Overall, this means that with $g_{A0} > 0$ and $\xi > \delta$ we would observe long-run stagnation (in both the semi-endogenous and fully endogenous models) only in the level of technology, and not in the level of output per capita, just when $\frac{\dot{H}_t}{H_t} = \xi + (\epsilon - 1)n - \delta < 0$. As a result, with respect to Jones (2022) –where there is no human capital accumulation– we conclude that, even accounting for the presence of class-size effects, the inclusion of (aggregate and, hence, per capita) human capital accumulation at positive rates allows preventing the empty planet outcome simultaneously in the level of real income per capita and in the level of technology. The inclusion of class-size effects does not exclude the possibility of observing technological stagnation (and only that!) just for a particular combination of parameter-values, namely when: $\frac{\dot{H}_t}{H_t} = \xi + (\epsilon - 1)n - \delta < 0$. The following sub-section analyzes numerically these conclusions.

2.2 Some Quantitative Results

Despite the introduction of positive (even though small, $0 < \epsilon < 1$) class size effects may lead, under a particular combination of parameter-values $\left[\xi + (\epsilon - 1)n - \delta < 0\right]$, to technological stagnation in the very long run, it is useful to analyze if, quantitatively, we could or could not ultimately expect such an outcome to occur in real life. To do this, we stick to Bloom et al.

(2020) and Jones (2022) and use the following parameter-values: $\beta = 3$; $\sigma = 1$; $\lambda = 1$; $n = 0.01$. Furthermore, from the Penn World Table 10.0 by Feenstra et al. (2015) for the last 70 years in the US, we see that the annual average growth rate of per capita human capital (g_{h0}) has been equal to 0.534%. In our model with class-size effects, this implies: $g_h < n \Leftrightarrow \xi + (\epsilon - 1)n - \delta < 0$. If we use $g_{y0} = 0.02$ (Jones, 2016, Fig. 1; Feenstra et al., 2015), then $g_{A0} = \frac{1}{\sigma}(g_{y0} - g_{h0}) = 0.01466$, a value not so different from the *'initial TFP growth rate of 1%'* of Jones (2022, p.10). With these parameter-values, using the two equations already obtained above when $g_{A0} > 0$, $\xi > \delta$, and $\xi + (\epsilon - 1)n - \delta < 0$, we get:

$$\lim_{t \to +\infty} \frac{A_t}{A_0} = \left[1 + 3\frac{0.01466}{0.00466}\right]^{1/3} = 2.1854$$

for the semi-endogenous growth model,

and:

$$\lim_{t \to +\infty} \frac{A_t}{A_0} = e^{\frac{0.01466}{0.00466}} = 23.2411$$

for the fully endogenous growth model. These two values are higher than those obtained by Jones (2022) – compare with values around 1.59 and 2.72, respectively.[9] The crucial difference across our and Jones (2022)'s models is

[9] To obtain these two numbers, use the values in Jones (2022, pp. 9-10) and plug them in his equations for A^* in Result 1 and Result 2 (Jones, 2022, pp. 7 and 9). One should note that the higher σ and the lower λ (consistently with Jones, 1995, and Alvarez-Peleaz and Groth, 2005), the higher the very long-run level of technology, A_t, above which technological growth would vanish (and stagnation of technology would occur).

that in our case the growth rate of output per capita is positive in the very long run and equal to the growth rate of human capital (0.534%).

Alternatively, we could have used the estimated class-size effect of $\epsilon = 0.1$ from e.g. Bandiera et al. (2010). Given a growth rate of per capita human capital of $\frac{\dot{h}_t}{h_t} = \xi + \epsilon n - \delta = 0.00534$, we have: $\xi - \delta = 0.00534 - 0.1n$.

Remember that in the model with *class-size effects* we would observe TFP stagnation at some point in time in the future when:

$$n > \frac{\xi - \delta}{1 - \epsilon} = \frac{0.00534 - 0.1n}{0.9} \qquad \Rightarrow \qquad n > 0.00534.$$

Notice that this lower-bound (0.534%) for the negative growth rate of population that guarantees future stagnation in the level of technology is lower than the value of 1% used by Jones (2022).

For different reasonable sets of parameters available in the literature, the following Table 1 presents the constant values of $\lim_{t \to +\infty} \frac{A_t}{A_0}$ in the semi-endogenous and fully endogenous growth models with the presence of class-size effects in the accumulation of per capita human capital.[10] All the scenarios considered assume the data-based values of: $n = 0.01$, $\frac{\dot{h}_t}{h_t} = \xi + \epsilon n - \delta = 0.00534$, and $g_{y0} = \sigma g_{A0} + \xi - \delta + \epsilon n = 0.02$.

[10] It is worth noting that while the normalization of σ (see Jones, 2022, Table 2, p. 27) is neutral for the long-run level of GDP per capita in the presence of declining population, the same normalization is not neutral for the level of technology (A_t), for which stagnation in the technological level occurs.

Results reported in Table 1 appear quite sensitive to oscillations of the spillover $(1-\beta)$, duplication (λ), and gains of specialization (σ) parameters in the semi-endogenous growth model, and only to the gains of specialization (σ) parameter in the fully endogenous growth model.[11]

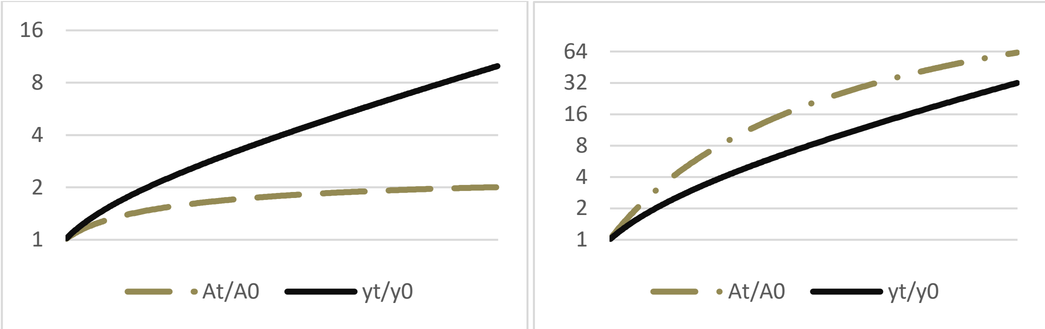
Table 1: *Technology Stagnation Results according to Different Calibrations*

Parameter Values		Source	$\lim_{t\to+\infty} \frac{A_t}{A_0}$	$\lim_{t\to+\infty} \frac{A_t}{A_0}$
			Semi-Endogenous Growth Model	Fully Endogenous Growth Model
$\beta = 0.271$ $\lambda = 0.5$	$\sigma = 0.196$	Jones and Williams (2000, Table 2)	4376, 55	9347560
$\beta = 0.417$ $\lambda = 0.75$	$\sigma = 0.196$	Jones and Williams (2000, Table 2)	245,56	9347560
$\beta = 0.2$ $\lambda = 0.2$	$\sigma = 0.5$	Sequeira and Neves (2018, 2019) and USPO	20615,2	540,15

Note: To infer the gains from specialization parameter σ in the last line we use data from USPO (Marco et al., 2015), calculating the annual growth rate of the stock of patents between 1950 and 2014.

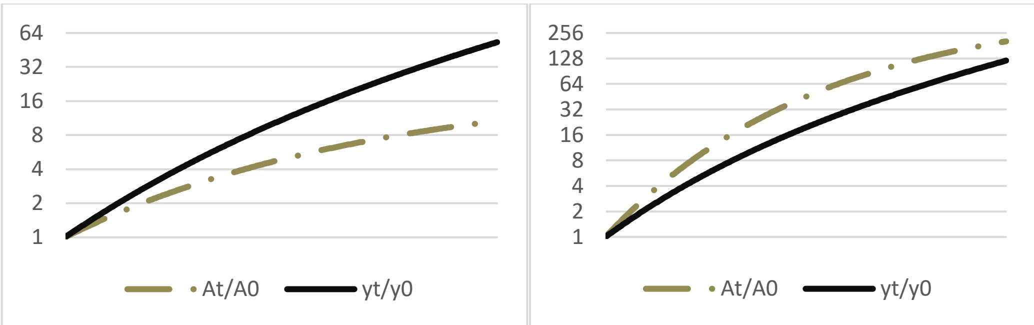
As a further exercise, we now simulate the series for the level of technology and the level of GDP per capita across time, using the baseline calibration values in Jones (2022) – panel (a) in the two Figures below - as well as our alternative calibration in the first line of Table 1 – panel (b) in the two Figures below.

[11] A feature common to results in Jones (2022).



(a) $\beta = 3;\ \sigma = 1;\ \lambda = 1;\ n = 1\%;\ g_{y0} = 2\%$ **(b)** $\beta = 0.271;\ \sigma = 0.196;\ \lambda = 0.5;\ n = 1\%;\ g_{y0} = 2\%$

Figure 1: Simulation of a Transitional Dynamics Path for two possible calibrated parameter sets in the semi-endogenous growth model. Note: the *y-axis* is in log-scale.



(a) $\beta = 3;\ \sigma = 1;\ \lambda = 1;\ n = 1\%;\ g_{y0} = 2\%$ **(b)** $\beta = 0.271;\ \sigma = 0.196;\ \lambda = 0.5;\ n = 1\%;\ g_{y0} = 2\%$

Figure 2: Simulation of a Transitional Dynamics Path for two possible calibrated parameter sets in the fully endogenous growth model. Note: the *y-axis* is in log-scale.

As expected, stagnation of TFP is faster in Figures 1a and 2a than in Figures 1b and 2b, while there is no stagnation in GDP per capita. Also, as expected, the TFP-stagnation result is more difficult to be seen in Figure 2 (fully endogenous model) than in Figure 1 (semi-endogenous model), which is also consistent with the theoretical results presented above and with results presented in Table 1.

A permanent positive shock in the birth rate or a permanent negative shock in the death rate can make the economy switch towards an equilibrium where even technology would not stagnate. This can be immediately seen by observing that a fall in the value of n may imply that the condition $\xi + \left(\epsilon - 1\right) n - \delta < 0$ is no longer fulfilled.

3. Conclusion

There are two main motivations for this paper. First, demographic data is showing that population decline will soon become a stylized fact for the most part of the World and it is already a fact in the most advanced ones. Then human capital has been pointed out as an important source of growth and productivity.

We depart from the recent contribution of Jones (2022) and introduce human capital in the technological knowledge (or R&D) production function. Our main result is that, once a class size effect in human capital accumulation is accounted for, it is possible to reconcile the TFP secular stagnation result with that of a positive long-run growth rate in GDP per capita.

This paper also opens prospects for future work. First, we intend to consider the case of endogenous fertility choices by agents, with the objective of studying the conditions under which the choice of low fertility-levels (below the replacement level) may be optimal, along with the influence of class size effects on this choice. Then, we may also consider the case in which there is a higher bound or high degree of obsolescence of human capital.

References

Aghion, P., and P. Howitt (1992). A Model of Growth through Creative Destruction. *Econometrica*, 60(2), 323-351.

Alvarez-Pelaez, M.J., and C. Groth (2005). Too little or too much R&D?. *European Economic Review*, *49*(2), 437-456.

Arnold, L.G. (1998). Growth, welfare, and trade in an integrated model of human-capital accumulation and research. *Journal of Macroeconomics*, 20(1), 81-105.

Bandiera, O., Larcinese, V., and I. Rasul (2010). Heterogeneous class size effects: New evidence from a panel of university students. *The Economic Journal*, *120*(549), 1365-1398.

Bils, M., and P.J. Klenow (2000). Does Schooling Cause Growth? *American Economic Review*, 90(5), 1160-1183

Bloom, N., Jones, C.I., Van Reenen, J., and M. Webb (2020). Are Ideas Getting Harder to Find? *American Economic Review*, 110(4), 1104-1144.

Brinca, P., Chari, V.V., Kehoe, P., McGratten, E., 2017. Accounting for business cycles. In: Uhlig, H., Taylor, J. (Eds.), *Handbook of Macroeconomics*, 2nd edition. Elsevier, Amsterdam

Bucci, A. (2022). *Can a Negative Population Growth Rate Sustain a Positive Economic Growth Rate in the Long-Run?* Manuscript Under Review.

Dinopoulos, E. , Thompson, P. (1999). Scale Effects in Schumpeterian Models of Economic Growth. Journal of Evolutionary Economics 9 (2), 157–185 .

Dinopoulos, E. , Thompson, P. (2000). Endogenous growth in a cross-section of countries. J. Int. Econ. 51, 335–362 .

Feenstra, R.C., Inklaar, R., and M.P. Timmer (2015). The Next Generation of the Penn World Table. American Economic Review, 105(10), 3150-3182, available for download at www.ggdc.net/pwt

Fernald, J. (2014). Productivity and potential output before, during, and after the Great Recession. *NBER Macroeconomics Annual* 29. National Bureau of Economic Research, Cambridge, Massachusetts .

Funke, M. and Strulik, H. (2000), "On Endogenous Growth with Physical Capital, Human Capital and Product Variety", *European Economic Review 44*, 491–515.

Grossman, G. and Helpman, E. (1991), *Innovation and Growth in the Global Economy*, MITPress, Cambridge, MA.

Jones, C.I. (1995). R&D-Based Models of Economic Growth. *Journal of Political Economy* 103 (4): 759–784.

Jones, C.I., and J.C. Williams (2000). Too much of a good thing? The economics of investment in R&D. *Journal of Economic Growth*, 5(1), 65-85.

Jones, C.I. (2016). The Facts of Economic Growth, Handbook of Macroeconomics, Chapter 1, Editor(s): J.B. Taylor, and H. Uhlig, Elsevier, Volume 2, Chapter 1, 3-69.

Jones, C.I. (2022). The end of economic growth? Unintended consequences of a declining population. *American Economic Review*, forthcoming. Available at: https://www.aeaweb.org/articles?id=10.1257/aer.20201605

Kasparov, G. , Levchin, M. , Thiel, P. (2012). The Blueprint: Reviving Innovation, Rediscovering Risk, and Rescuing the Free Market. W. W. Norton & Company .

Konstantopoulos, S. and V. Chung (2009). What Are the Long-Term Effects of Small Classes on the Achievement Gap? Evidence from the Lasting Benefits Study. *American Journal of Education*, 116(1): 125-154

Marco, A.C., Carley, M., Jackson, S., and A.F. Myers (2015). The USPTO Historical Patent Data Files: Two centuries of invention, U.S. Patent and Trademark Office, WP No. 2015-1. Available at http://www.uspto.gov/economics. Data available at. http://www.uspto.gov/learning-and-resources/electronic - data- products / historical-patent-data-files.

Mathis, W. (2017). The Effectiveness of Class Size Reduction. *Psychosociological Issues in Human Resource Management* 5(1): 176–183.

Rebelo, S. (1991). Long-Run Policy Analysis and Long-Run Growth. *Journal of Political Economy*, 99(3): 500-521

Romer, P.M. (1990). Endogenous technological change. *Journal of Political Economy* 98 (5), 71–102.

Sequeira, T., P. Gil, O. Afonso (2018). Endogenous Growth and Entropy. *Journal of Economic Behavior and Organization*, 154: 100-120

World Bank (2021). The Human Capital Index 2020 Update: Human Capital in the Time of COVID-19. Washington, DC: World Bank. doi:10.1596/978-1-4648-1552-2. License: Creative Commons Attribution CC BY 3.0 IGO.

APPENDICES

A. Derivation of Propositions 1 and 2 (exogenous human capital accumulation and negative and exogenous population growth)

PROOF OF PROPOSITION 1

(a) Semi-endogenous growth model ($\beta > 0$ and $0 < \lambda < 1$ in Eq. 3):

Integrate the differential equation:

$$\int A_t^{\beta-1} dA_t = \int \alpha \left(N_t h_t\right)^{\lambda} dt .$$

This gives:

$$\frac{1}{\beta} A_t^{\beta} = \alpha \left(N_0 h_0\right)^{\lambda} \int \left[e^{-nt} e^{(\xi-\delta+n)t}\right]^{\lambda} dt \qquad \Rightarrow$$

$$A_t^{\beta} = \frac{\alpha\beta \left(N_0 h_0\right)^{\lambda} e^{\lambda(\xi-\delta)t}}{\lambda\left(\xi-\delta\right)} + C_0$$

By setting t=0 we can find the constant term C_0 as:

$$C_0 = A_0^{\beta} - \frac{\alpha\beta\left(N_0 h_0\right)^{\lambda}}{\lambda\left(\xi-\delta\right)}$$

If C_0 is replaced in the expression for A_t^{β} written above, after some algebra, it is possible to conclude that:

$$\frac{A_t}{A_0} = \left\{1 - \frac{\beta\alpha\left(N_0 h_0\right)^{\lambda} A_0^{-\beta}}{\lambda\left(\xi-\delta\right)}\left[1 - e^{\lambda(\xi-\delta)t}\right]\right\}^{1/\beta} .$$

Finally, by using $g_{A0} = \alpha\left(N_0 h_0\right)^{\lambda} A_0^{-\beta}$, we get Eq. (4) in the text:

$$\frac{A_t}{A_0} = \left\{1 - \frac{\beta g_{A0}}{\lambda\left(\xi-\delta\right)}\left[1 - e^{\lambda(\xi-\delta)t}\right]\right\}^{1/\beta} .$$

Using Eq. (1) in the text gives:

$$\frac{y_t}{y_0} = \left(\frac{A_t}{A_0}\right)^{\sigma} \frac{h_t}{h_0}$$

By substituting in the above expression $\frac{h_t}{h_0} = e^{(\xi-\delta+n)t}$, and the ratio $\frac{A_t}{A_0}$ from Eq. (4) in the text we get:

$$\frac{y_t}{y_0} = \left\{1 - \frac{\beta}{\lambda}\left(\frac{g_{A0}}{\xi - \delta}\right)\left[1 - e^{\lambda(\xi-\delta)t}\right]\right\}^{\sigma/\beta} \cdot e^{(\xi-\delta+n)t}$$

As a final step, we now replace in the above expression $g_{y0} = \sigma g_{A0} + \xi - \delta + n$. This allows us obtaining the following expression, which is Eq. (5) in the text:

$$\frac{y_t}{y_0} = \left\{1 - \frac{\beta}{\sigma\lambda}\left(\frac{g_{y0} - n}{\xi - \delta} - 1\right)\left[1 - e^{\lambda(\xi-\delta)t}\right]\right\}^{\sigma/\beta} \cdot e^{(\xi-\delta+n)t} \quad \blacksquare$$

(b) Fully-endogenous growth model ($\beta = 0$ and $\lambda = 1$ in Eq. 3):

By setting $\beta = 0$ and $\lambda = 1$ in Eq. (3) in the text, the law of motion of the stock of ideas (A_t) is transformed into: $\dot{A}_t = \alpha A_t N_t h_t$.

Integrate the differential equation:

$$\begin{aligned}\int \frac{dA_t}{A_t} &= \int \alpha\left(N_t h_t\right) dt \\ &= \alpha\left(N_0 h_0\right)\int e^{-nt} \cdot e^{(\xi-\delta+n)t} dt \\ &= \alpha\left(N_0 h_0\right)\int e^{(\xi-\delta)t} dt\end{aligned}$$

which gives:

$$\log A_t = \frac{\alpha N_0 h_0}{(\xi - \delta)} e^{(\xi-\delta)t} + C_0$$

By setting t=0, we can immediately find the constant term C_0 as:

$$C_0 = \log A_0 - \frac{\alpha N_0 h_0}{(\xi - \delta)}.$$

If we use $g_{A0} = \alpha N_0 h_0$ in the above expression and replace it in the expression for $\log A_t$, we finally get:

$$\log A_t = \log A_0 - \frac{g_{A0}}{(\xi - \delta)}\left[1 - e^{(\xi-\delta)t}\right].$$

From this expression we obtain immediately Eq. (6) in the text:

$$\frac{A_t}{A_0} = e^{\frac{g_{A0}}{(\xi-\delta)}\left[e^{(\xi-\delta)t} - 1\right]}.$$

By using Eq. (1) in the text, we can express the output differences in per capita terms as follows:

$$\frac{y_t}{y_0} = \left(\frac{A_t}{A_0}\right)^{\sigma} \cdot \left(\frac{h_t}{h_0}\right).$$

By plugging in the above expression $\frac{h_t}{h_0} = e^{(\xi-\delta+n)t}$ and $\frac{A_t}{A_0}$ from Eq. (6) in the text, we get Eq. (7) in the text:

$$\frac{y_t}{y_0} = \left\{ e^{\frac{1}{\sigma}\left(\frac{g_{y0}-n}{\xi-\delta}-1\right)\left[e^{(\xi-\delta)t}-1\right]} \right\}^{\sigma} \cdot e^{(\xi-\delta+n)t}, \qquad g_{y0} = \sigma g_{A0} + \xi - \delta + n \quad \blacksquare$$

<u>PROOF OF PROPOSITION 2</u>

(a) Semi-endogenous growth model ($\beta > 0$ and $0 < \lambda < 1$ in Eq. 3):

Integrate the differential equation:

$$\int A_t^{\beta-1} dA_t = \int \alpha \left(N_t h_t\right)^{\lambda} dt,$$

which gives

$$\frac{1}{\beta} A_t^{\beta} = \alpha\left(N_0 h_0\right)^{\lambda} \int \left[e^{-nt} \cdot e^{(\xi+\epsilon n-\delta)t}\right]^{\lambda} dt \qquad \Rightarrow$$

$$A_t^{\beta} = \frac{\alpha\beta\left(N_0 h_0\right)^{\lambda} e^{\lambda\left[\xi+(\epsilon-1)n-\delta\right]t}}{\lambda\left[\xi+(\epsilon-1)n-\delta\right]} + C_0$$

By setting $t = 0$ to solve for the constant C_0 gives

$$C_0 = A_0^{\beta} - \frac{\alpha\beta\left(N_0 h_0\right)^{\lambda}}{\lambda\left[\xi+(\epsilon-1)n-\delta\right]}$$

If the constant C_0 is replaced in the general solution of the differential equation for the level of technology, we finally get:

$$\frac{A_t}{A_0} = \left\{1 - \frac{\beta\alpha\left(N_0 h_0\right)^{\lambda} A_0^{-\beta}}{\lambda\left[\xi+(\epsilon-1)n-\delta\right]} \cdot \left[1 - e^{\lambda\left[\xi+(\epsilon-1)n-\delta\right]t}\right]\right\}^{1/\beta}$$

By using $g_{A0} = \alpha\left(N_0 h_0\right)^{\lambda} A_0^{-\beta}$, we get eq. (9) in the main text:

$$\frac{A_t}{A_0} = \left\{1 - \frac{\beta g_{A0}}{\lambda\left[\xi+(\epsilon-1)n-\delta\right]} \cdot \left[1 - e^{\lambda\left[\xi+(\epsilon-1)n-\delta\right]t}\right]\right\}^{1/\beta}.$$

By using Eq. (1) in the main text we can express per-capita output differences as:

$$\frac{y_t}{y_0} = \left(\frac{A_t}{A_0}\right)^{\sigma} \frac{h_t}{h_0}.$$

By plugging in the equation above $\frac{h_t}{h_0} = e^{(\xi+\epsilon n-\delta)t}$ and the differences in the stock of knowledge $\frac{A_t}{A_0}$ obtained earlier, we get:

$$\frac{y_t}{y_0} = \left\{1 - \frac{\beta}{\lambda}\left[\frac{g_{A0}}{\xi + (\epsilon - 1)n - \delta}\right]\left[1 - e^{\lambda\left[\xi+(\epsilon-1)n-\delta\right]t}\right]\right\}^{\sigma/\beta} \cdot e^{(\xi+\epsilon n-\delta)t}$$

Finally, by using in the last expression $g_{y0} = \sigma g_{A0} + \xi - \delta + \epsilon n$, we get the following equation, namely Eq. (10) in the text:

$$\frac{y_t}{y_0} = \left\{1 - \frac{\beta}{\sigma\lambda}\left[\frac{g_{y0} - n}{\xi + (\epsilon - 1)n - \delta} - 1\right]\left[1 - e^{\lambda\left[\xi+(\epsilon-1)n-\delta\right]t}\right]\right\}^{\sigma/\beta} \cdot e^{(\xi+\epsilon n-\delta)t} \quad \blacksquare$$

(b) Fully endogenous growth model ($\beta = 0$ and $\lambda = 1$ in Eq. 3):

By setting $\beta = 0$ and $\lambda = 1$ in Eq. (3) in the text, the law of motion of the stock of ideas (A_t) is transformed into: $\dot{A}_t = \alpha A_t N_t h_t$.

Integrate the differential equation:

$$\begin{aligned}\int \frac{dA_t}{A_t} &= \int \alpha\left(N_t h_t\right) dt \\ &= \alpha\left(N_0 h_0\right)\int e^{-nt} \cdot e^{(\xi+\epsilon n-\delta)t} dt \\ &= \alpha\left(N_0 h_0\right)\int e^{\left[(\epsilon-1)n+\xi-\delta\right]t} dt\end{aligned}$$

which gives:

$$\log A_t = \frac{\alpha\left(N_0 h_0\right) \cdot e^{\left[\xi+(\epsilon-1)n-\delta\right]t}}{\xi + (\epsilon - 1)n - \delta} + C_0$$

By setting t=0, we can find the constant term C_0 as:

$$C_0 = \log A_0 - \frac{\alpha\left(N_0 h_0\right)}{\xi + (\epsilon - 1)n - \delta}$$

If we use $g_{A0} = \alpha N_0 h_0$ in the above expression and replace it in the solution of the differential equation, we obtain:

$$\log A_t = \log A_0 - \frac{g_{A0}}{\xi + (\epsilon - 1)n - \delta}\left[1 - e^{[\xi + (\epsilon-1)n - \delta]t}\right]$$

From this expression we obtain immediately Eq. (11) in the text:

$$\frac{A_t}{A_0} = e^{\frac{g_{A0}}{\xi + (\epsilon-1)n - \delta}\left[e^{[\xi + (\epsilon-1)n - \delta]t} - 1\right]}$$

By using Eq. (1) in the text, we can express the per-capita output differences as follows:

$$\frac{y_t}{y_0} = \left(\frac{A_t}{A_0}\right)^{\sigma} \frac{h_t}{h_0}$$

By plugging in the above expression $\frac{h_t}{h_0} = e^{(\xi + \epsilon n - \delta)t}$ and the differences in the stock of knowledge $\frac{A_t}{A_0}$ from Eq. (11) we get Eq. (12) in the text:

$$\frac{y_t}{y_0} = \left\{ e^{\frac{1}{\sigma}\left[\frac{g_{y0} - n}{\xi + (\epsilon-1)n - \delta} - 1\right] \cdot \left[e^{[\xi + (\epsilon-1)n - \delta]t} - 1\right]} \right\}^{\sigma} \cdot e^{(\xi + \epsilon n - \delta)t}, \qquad g_{y0} = \sigma g_{A0} + \xi - \delta + \epsilon n \qquad \blacksquare$$

B. Derivation of results when $\xi = \delta$ (models with exogenous human capital accumulation, negative and exogenous population growth, and with or without *class-size effects*)

We start by analyzing what happens in the **basic setup <u>without</u> *class-size effects* in per capita human capital accumulation** (Eqs. 1-2-3 in the text). In this framework, when $\xi = \delta$, the law of motion of human capital per capita becomes:

$$\dot{h}_t = n h_t .$$

Below we provide results and derivations for the fully endogenous and the semi-endogenous models in this particular framework.

(a) Semi-endogenous growth model ($\beta > 0$ and $0 < \lambda < 1$ in Eq. 3):

Integrate the differential equation:

$$\int A_t^{\beta-1} dA_t = \int \alpha \left(N_t h_t\right)^{\lambda} dt .$$

This gives:

$$\frac{1}{\beta}A_t^{\beta} = \alpha\left(N_0h_0\right)^{\lambda}\int\left[e^{-nt}\cdot e^{nt}\right]^{\lambda}dt \qquad \Rightarrow$$

$$A_t^{\beta} = \alpha\beta\left(N_0h_0\right)^{\lambda}t + C_0$$

By setting t=0 we can find the constant term C_0 as:

$$C_0 = A_0^{\beta}$$

If C_0 is replaced in the expression for A_t^{β} written above, it is possible to conclude that:

$$\frac{A_t}{A_0} = \left[1 + \beta\alpha\left(N_0h_0\right)^{\lambda}\cdot t\cdot A_0^{-\beta}\right]^{1/\beta}.$$

Finally, by using $g_{A0} = \alpha\left(N_0h_0\right)^{\lambda}A_0^{-\beta}$, we get:

$$\frac{A_t}{A_0} = \left(1 + \beta\cdot g_{A0}\cdot t\right)^{1/\beta}.$$

Using Eq. (1) in the text gives:

$$\frac{y_t}{y_0} = \left(\frac{A_t}{A_0}\right)^{\sigma}\frac{h_t}{h_0}$$

By substituting in the above expression $\frac{h_t}{h_0} = e^{nt}$, and the ratio $\frac{A_t}{A_0} = \left(1 + \beta\cdot g_{A0}\cdot t\right)^{1/\beta}$ we get:

$$\frac{y_t}{y_0} = \left(1 + \beta\cdot g_{A0}\cdot t\right)^{\sigma/\beta}\cdot e^{nt}$$

As a final step, we can replace in the above expression: $g_{y0} = \sigma g_{A0} + n$. This allows us recasting y_t / y_0 as:

$$\frac{y_t}{y_0} = \left[1 + \frac{\beta}{\sigma}\left(g_{y0} - n\right)\cdot t\right]^{\sigma/\beta}\cdot e^{nt} \qquad \blacksquare$$

(b) Fully endogenous growth model ($\beta = 0$ and $\lambda = 1$ in Eq. 3):

By setting $\beta = 0$ and $\lambda = 1$ in Eq. (3) in the text, the law of motion of the stock of ideas (A_t) is transformed into: $\dot{A}_t = \alpha A_t N_t h_t$.

Integrate the differential equation:

$$\int \frac{dA_t}{A_t} = \int \alpha\left(N_t h_t\right) dt$$
$$= \alpha\left(N_0 h_0\right) \int e^{-nt} \cdot e^{nt} dt$$
$$= \alpha\left(N_0 h_0\right) \int dt$$

which gives:

$$\log A_t = \alpha\left(N_0 h_0\right) t + C_0$$

By setting t=0, we can immediately find the constant term C_0 as:

$$C_0 = \log A_0 .$$

If we use $g_{A0} = \alpha\left(N_0 h_0\right)$ and replace it in the expression for $\log A_t$ above, we finally get:

$$\log A_t = g_{A0} \cdot t + \log A_0$$

From this expression we obtain immediately:

$$\frac{A_t}{A_0} = e^{g_{A0} \cdot t} .$$

By using Eq. (1) in the text, we can express the output differences in per capita terms as follows:

$$\frac{y_t}{y_0} = \left(\frac{A_t}{A_0}\right)^{\sigma} \cdot \left(\frac{h_t}{h_0}\right).$$

By plugging in the above expression $\frac{h_t}{h_0} = e^{nt}$ and $\frac{A_t}{A_0} = e^{g_{A0} \cdot t}$, we obtain

$$\frac{y_t}{y_0} = e^{(\sigma g_{A0} + n)t} = e^{g_{y0} \cdot t}, \qquad g_{y0} = \sigma g_{A0} + n \qquad \blacksquare$$

We now analyze what happens in the **basic setup <u>augmented with</u> the presence of *class-size effects* in per capita human capital accumulation** (Eqs. 1-3-8 in the text). In this framework, when $\xi = \delta$, the law of motion of human capital per capita becomes:

$$\dot{h}_t = \epsilon n h_t .$$

Below we provide results and derivations for the fully endogenous and the semi-endogenous models in this particular framework.

(a) Semi-endogenous growth model ($\beta > 0$ and $0 < \lambda < 1$ in Eq. 3):

Integrate the differential equation:

$$\int A_t^{\beta-1} dA_t = \int \alpha \left(N_t h_t\right)^{\lambda} dt$$

This gives:

$$\frac{1}{\beta} A_t^{\beta} = \alpha \left(N_0 h_0\right)^{\lambda} \int \left(e^{-nt} \cdot e^{\epsilon nt}\right)^{\lambda} dt \qquad \Rightarrow$$

$$A_t^{\beta} = \frac{\alpha\beta \left(N_0 h_0\right)^{\lambda} \cdot e^{\lambda(\epsilon-1)nt}}{\lambda\left(\epsilon-1\right)n} + C_0$$

By setting t=0 we can find the constant term C_0 as:

$$C_0 = A_0^{\beta} - \frac{\alpha\beta \left(N_0 h_0\right)^{\lambda}}{\lambda\left(\epsilon-1\right)n}$$

If C_0 is replaced in the expression for A_t^{β} written above, it is possible to conclude that:

$$\frac{A_t}{A_0} = \left\{1 - \frac{\beta\alpha \left(N_0 h_0\right)^{\lambda} A_0^{-\beta}}{\lambda\left(\epsilon-1\right)n} \cdot \left[1 - e^{\lambda(\epsilon-1)nt}\right]\right\}^{1/\beta}.$$

Finally, by using $g_{A0} = \alpha \left(N_0 h_0\right)^{\lambda} A_0^{-\beta}$, we get:

$$\frac{A_t}{A_0} = \left\{1 - \frac{\beta g_{A0}}{\lambda\left(\epsilon-1\right)n} \cdot \left[1 - e^{\lambda(\epsilon-1)nt}\right]\right\}^{1/\beta}.$$

Using Eq. (1) in the text gives:

$$\frac{y_t}{y_0} = \left(\frac{A_t}{A_0}\right)^{\sigma} \cdot \left(\frac{h_t}{h_0}\right)$$

By substituting in the above expression $\frac{h_t}{h_0} = e^{\epsilon nt}$, and the ratio $\frac{A_t}{A_0} = \left\{1 - \frac{\beta g_{A0}}{\lambda\left(\epsilon-1\right)n} \cdot \left[1 - e^{\lambda(\epsilon-1)nt}\right]\right\}^{1/\beta}$ we get:

$$\frac{y_t}{y_0} = \left\{1 - \frac{\beta}{\lambda} \cdot \left[\frac{g_{A0}}{\left(\epsilon-1\right)n}\right] \cdot \left[1 - e^{\lambda(\epsilon-1)nt}\right]\right\}^{\sigma/\beta} \cdot e^{\epsilon nt}$$

As a final step, we can replace in the above expression: $g_{y0} = \sigma g_{A0} + \epsilon n$. This allows us recasting y_t / y_0 as:

$$\frac{y_t}{y_0} = \left\{1 - \frac{\beta}{\sigma\lambda} \cdot \left[\frac{g_{y0} - n}{(\epsilon - 1)n} - 1\right] \cdot \left[1 - e^{\lambda(\epsilon-1)nt}\right]\right\}^{\sigma/\beta} \cdot e^{\epsilon nt} \quad \blacksquare$$

(b) Fully endogenous growth model ($\beta = 0$ and $\lambda = 1$ in Eq. 3):

By setting $\beta = 0$ and $\lambda = 1$ in Eq. (3) in the text, the law of motion of the stock of ideas (A_t) is transformed into: $\dot{A}_t = \alpha A_t (N_t h_t)$.

Integrate the differential equation:

$$\int \frac{dA_t}{A_t} = \int \alpha (N_t h_t) dt$$

$$= \alpha (N_0 h_0) \int e^{-nt} \cdot e^{\epsilon nt} dt$$

$$= \frac{\alpha (N_0 h_0)}{(\epsilon - 1)n} \cdot e^{(\epsilon-1)nt}$$

which, in the end, gives:

$$\log A_t = \frac{\alpha (N_0 h_0)}{(\epsilon - 1)n} e^{(\epsilon-1)nt} + C_0$$

By setting t=0, we can immediately find the constant term C_0 as:

$$C_0 = \log A_0 - \frac{\alpha (N_0 h_0)}{(\epsilon - 1)n}.$$

If we use $g_{A0} = \alpha (N_0 h_0)$ and replace it in the expression for $\log A_t$ above, we finally get:

$$\log A_t = \log A_0 - \frac{g_{A0}}{(\epsilon - 1)n} \cdot \left[1 - e^{(\epsilon-1)nt}\right]$$

From this expression we obtain:

$$\frac{A_t}{A_0} = e^{\frac{g_{A0}}{(\epsilon-1)n} \cdot \left[e^{(\epsilon-1)nt} - 1\right]}.$$

By using Eq. (1) in the text, we can express the output differences in per capita terms as follows:

$$\frac{y_t}{y_0} = \left(\frac{A_t}{A_0}\right)^{\sigma} \cdot \left(\frac{h_t}{h_0}\right).$$

By plugging in the above expression $\frac{h_t}{h_0} = e^{\epsilon nt}$ and $\frac{A_t}{A_0} = e^{\frac{g_{A0}}{(\epsilon-1)n} \cdot \left[e^{(\epsilon-1)nt} - 1\right]}$, we obtain

$$\frac{y_t}{y_0} = e^{\frac{\sigma g_{A0}}{(\epsilon-1)n}\cdot\left[e^{(\epsilon-1)nt}-1\right]} \cdot e^{\epsilon nt},$$

or, alternatively: $$\frac{y_t}{y_0} = \left\{ e^{\frac{1}{\sigma}\cdot\left[\frac{g_{y0}-n}{(\epsilon-1)n}-1\right]\cdot\left[e^{(\epsilon-1)nt}-1\right]} \right\}^{\sigma} \cdot e^{\epsilon nt}, \qquad g_{y0} = \sigma g_{A0} + \epsilon n \qquad \blacksquare$$