MEAN-VARIANCE EFFICIENCY VERSUS POSITIVE SKEWNESS SEEKING IN PORTFOLIO SELECTION

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MEAN-VARIANCE EFFICIENCY VERSUS POSITIVE SKEWNESS SEEKING IN PORTFOLIO SELECTION

(Preliminary version)

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Abstract

This paper compares the performance of efficient portfolios based on the Markowitz (1952) mean-variance model with portfolios with high skewness. The assumption that the return of assets follows the normal distribution is the basis of the mean-variance model. However, the return of financial assets often deviates from the normal distribution, namely due to positive skewness, which may offer some advantages to investors. Previous literature reports several obstacles that make it difficult to include skewness in portfolio optimization, and that there is a trade-off between return and skewness maximization. Mean-variance optimization versus the search for positive skewness is addressed in this paper by estimating comparative performance ratios between mean-variance efficient portfolios and portfolios with high skewness. The paper also estimates *probit* models which highlight the probability of obtaining higher return from portfolios with high skewness than from mean-variance optimized portfolios. The probability given by our estimations is, in general, relatively low, which suggests that mean-variance optimization must be preferred to the search for positive skewness as method of portfolio choice.

Keywords: Efficient frontier; Mean-variance optimization; Portfolio selection; Skewness.

JEL Classification: G10,

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1. Introduction

Modern financial theory began with the seminal article of Markowitz (1952), which proposed the mean-variance optimization model as the fundamental framework of portfolio choice. A significant number of papers in literature is devoted to efficient portfolio selection, which comprise 19000 citations of Markowitz's paper, reported by Kolm *et al.* (2014). Several issues about the original Markowitz model have been taken into consideration by literature, namely the inclusion of constraints in portfolio construction which, as Markowitz (1987) and Kolm *et al.* (2014) refer, can originate sup-optimal portfolios. Regime shifts in asset return distributions, which occur frequently and can disturb the process of portfolio choice based on the Markowitz model, were object of a dynamic version of the model, proposed by Bae *et al.* (2014).

One of the fundamental assumptions that underlies the mean-variance model is that asset returns follow the normal distribution. Failure to verify this assumption, namely asymmetric distribution of asset returns and its consequences for portfolio selection are the object of this paper. Portfolio selection in the presence of positive skewness, received attention of several authors, such as Kane (1982), who uses the Taylor expansion of the von Neumann-Morgenstern utility function to show that skewness preference reduces the risk premium requested by risk averter investors. Moreover, Kane(1982) underlines that the return bias to the right of expected return represented by positive skewness makes portfolios attractive to investors with prudent behaviour. Following in the same direction as Kane's conclusion, Chunhachinda *et al.*(1997) demonstrate that positive skewness makes portfolio selection a multi-objective problem, where investors seek, simultaneously, to maximize return and skewness and to minimize variance. However, De Athayde and Flôres (2004) show that there is a trade-off between expected return and skewness if both objectives are included in portfolio optimization, and these authors also underline that skewness constraints hamper the computing process, given the large dimension of the co-skewness matrix, which is equal to N^3 (N being the number of assets). Moreover, as Jiang *et al.* (2016) show, the inclusion of skewness constraint can cause loss in mean-variance efficiency. Given the computing difficulties caused by inclusion of skewness in the mean-variance portfolio selection process, Briec *et al.* (2016) propose a method for ranking portfolios based on a three-dimensional distance of expected return, variance, and skewness, relative to a benchmark portfolio. This paper compares mean-variance portfolio optimization with skewness seeking, using two alternate methods. First, we calculate comparative performance ratios between mean-variance efficient portfolios and portfolios with high skewness. The performance ratios used in this method are based on the performance ratios of Israelson (2004) and Sharpe (1966, 1994). Secondly, we use *probit* models to estimate the probability of obtaining higher return from portfolios with high skewness than from mean-variance efficient portfolios. The rest of the paper is organised as follows. Section 2 discusses the trade-off between mean-variance optimization and skewness maximization, and presents the methods used in the paper to compare mean-variance efficient portfolios with portfolios with high skewness. Section 3 discusses the empirical portfolio selection using a database of equity share returns from forty-four German firms, with monthly data from the beginning of 1999 to the end of 2019, comprising 252 observations of each individual series. Efficient portfolios based on the mean-variance model and portfolios with high skewness are subject to pairwise comparison by the performance comparison ratios and probability estimations, referred above. Section 4 sets out the conclusions.

2. Optimal portfolio selection under the mean-variance framework and seeking for skewness

2.1. The mean-variance model and the efficient frontier

The literature on portfolio optimization commonly accepts that, where there is uncertainty, investors wealth utility, $U(W)$ is governed by the paradigm of maximization of the von Neumann-Morgenstern utility function. According to this paradigm, the expected utility of the final wealth can be represented by a Taylor expansion around the utility of the expected wealth, as follows:

$$E\left[U(W)\right]=U(\bar{W})+U'(\bar{W})E(W-\bar{W})+\frac{1}{2}U''(\bar{W})E\left[(W-\bar{W})^2\right] + \frac{1}{6}U'''(\bar{W})E\left[(W-\bar{W})^3\right]+\ldots \quad (1)$$

Expected utility can be represented as a function of wealth variance and skewness, taking into consideration that:

$$E(W-\bar{W})=0 \quad (2)$$

the wealth variance is:

$$\sigma_W^2=E\left[(W-\bar{W})^2\right] \quad (3)$$

and the third-order moment, $E\left[(W-\bar{W})^3\right]$, is strictly related to skewness, Sk_W, since

$$Sk_W=\frac{N^2}{(N-1)(N-2)}\frac{E\left[(W-\bar{W})^3\right]}{\sigma_W^3} \quad (4)$$

N, being the number of observations. If, additionally, we define $Sk_W^*=E\left[(W-\bar{W})^3\right]$, the Taylor expansion of expected utility takes the following alternate representation:

$$E\left[U(W)\right]=U(\bar{W})+\frac{1}{2}U''(\bar{W})\sigma_W^2+\frac{1}{6}U'''(\bar{W})Sk_W^* \quad (5)$$

The second order derivative of the risk averter's utility function, $U''(W)$, is negative, implying that variance contributes negatively to utility, while the third order derivative, $U'''(W)$, is positive, implying that skewness contributes positively to utility.

The Markowitz (1952) mean-variance model neglects skewness, and the utility function to be maximized depends only on the portfolios' expected return, $E(R_P)$ and variance, σ_P^2. Markowitz (1959, 1987) suggests that the utility function of a risk averter can be represented by quadratic functions of wealth or return, from which results the following dependence of the utility function on the expected return and variance:

$$\begin{aligned} &U\left[E(R_P),\sigma_P^2\right]=\phi E(R_P)-\varphi\sigma_P^2 \\ &\phi>0;\varphi>0 \end{aligned} \quad (6)$$

or, alternately:

$$U\left[E(R_P),\sigma_P^2\right]=E(R_P)-\theta\sigma_P^2 \quad (7)$$

where $\theta = \varphi / \phi$, which is more suitable for quadratic programming than the previous representation. The common portfolio is composed of N risky assets existing in the economy, whose expected returns are represented as $E(R_i)$, i=1,…N, plus the risk-free asset, whose rate of return is r_f. Hence the expected return of a portfolio is the sum of the expected returns of the individual assets, weighted by the corresponding proportions in the portfolio, x_i, i=1,…, N, *i.e.*:

$$E(R_P)=\sum_{i=1}^{N}E(R_i)x_i+\left(1-\sum_{i=1}^{N}x_i\right)r_f \quad (8)$$

Portfolio variance, σ_P^2, is given by:

$$\sigma_P^2=\sum_{i=1}^{N}\sigma_i^2x_i^2+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\sigma_{ij}x_ix_j \quad (9)$$

where $\sigma_i^2=E\left[\left(R_i-E(R_i)\right)^2\right]$ is the variance of risky asset i, and $\sigma_{ij}=E\left[\left(R_i-E(R_i)\right)\left(R_j-E(R_j)\right)\right]$ is the covariance between risky assets i and j. Replacing $E(R_P)$ and σ_P^2 in the utility function represented in (6) according to their representations in (8) and (9), respectively, gives the following representation to the utility function:

$$U\left[E\left(R_P\right),\sigma_P^2\right]=\sum_{i=1}^{N}E\left(R_i\right)x_i+\left(1-\sum_{i=1}^{N}x_i\right)r_f-\theta\left[\sum_{i=1}^{N}\sigma_i^2x_i^2+\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\sigma_{ij}x_ix_j\right] \quad (10)$$

To maximize this utility function, their first order partial derivatives relative to the proportions x_i (i=1,…,N) must be equal to zero, which gives the system of N equations represented as follows:

$$\begin{aligned}
E\left(R_1\right)-r_f &= \sigma_1^2 z_1+\ldots+\sigma_{1i}z_i+\ldots+\sigma_{1N}z_N \\
&\ldots \\
E\left(R_i\right)-r_f &= \sigma_{1i} z_1+\ldots+\sigma_i^2 z_i+\ldots+\sigma_{iN}z_N \\
&\ldots \\
E\left(R_N\right)-r_f &= \sigma_{1N} z_1+\ldots+\sigma_{iN} z_i+\ldots+\sigma_N^2 z_N
\end{aligned} \quad (11),$$

where $z_i = 2\theta x_i$, i=1,…, N

After solving the system of equations represented in (11) to obtain z_1 ,…, z_i,…, z_N, the composition of the optimum portfolio, comprising N risky assets, is given by the proportions: $x_i = {z_i}\Big/{\sum_{k=1}^{N} z_k}$, i=1,…,N, which implies $\sum_{i=1}^{N} x_i = 1$. The portfolio of risky assets with this composition is represented by point M in Figure 1, where the capital market line (CML) is tangent to the efficient frontier (EF).

Insert Figure 1 about here

Each point of the capital market line represents the expected return and variance of a portfolio combining efficient portfolio *M* (commonly designated in literature as market portfolio) with the risk-free asset, whose interest rate, r_f, is the intercept of CML with the vertical axis. Repeating the calculations described above, using an arbitrary value r_a , instead of the risk-free interest rate, r_f, in the system of equations (11), we can determine the composition of another portfolio over the efficient frontier, represented as EFP_a in Figure 1. After having determined the compositions of two portfolios, *a* and *b*, over the efficient frontier, represented by their vectors of proportions, respectively

$\left[X^{(a)}\right]=\left[x_1^{(a)},...,x_i^{(a)},...,x_N^{(a)}\right]$ and $\left[X^{(b)}\right]=\left[x_1^{(b)},...,x_i^{(b)},...,x_N^{(b)}\right]$, the composition of every other portfolio *c*, over the efficient frontier, can be obtained through the linear combination of vectors $\left[X^{(a)}\right]$ and $\left[X^{(b)}\right]$, as follows:

$$\left[X^{(c)}\right]=\alpha\left[X^{(a)}\right]+(1-\alpha)\left[X^{(b)}\right] \quad (12)$$

2.2 The trade-off between mean-variance optimization and skewness maximization

Combining mean-variance with skewness in a single model of portfolio choice raises a complex, three-dimensional problem for portfolio optimization, where the purpose of investors is to maximize the following type of utility function:

$$\begin{aligned} &U\left[E(R_P),\sigma_P^2,Sk_P\right]=\phi E(R_P)-\varphi\sigma_P^2+\gamma Sk_P \\ &\phi>0;\varphi>0;\gamma>0 \end{aligned} \quad (13)$$

where positive skewness $Sk_P>0$ is added to the mean-variance utility function. The maximization of the three-dimensional utility function represented above raises several problems reported by De Athayde and Flôres (2004), the first of all being the program computing difficulties caused by the large dimension of the co-skewness matrix, which makes it very difficult to include in an optimization program. Given the number of assets N, the size of the co-skewness matrix is N^3, where the total number of distinct values is given by the total combinations of three elements with repetition out of N, *i.e.* $\binom{N+2}{3}$. The general element of the co-skewness matrix is $CoSk_{ijk}=E\left[\left(R_i-E(R_i)\right)\left(R_j-E(R_j)\right)\left(R_k-E(R_k)\right)\right]$, *i,j,k=1,...,N* . Specific cases of the co-skewness matrix are $Co_{iii}=E\left[\left(R_i-E(R_i)\right)^3\right]$, and $CoSk_{iij}=E\left[\left(R_i-E(R_i)\right)^2\left(R_j-E(R_j)\right)\right]$. Another difficulty in including expected return, variance, and skewness, together, in portfolio optimization, reported by De Athayde and Flôres (2004), is the absence of a complete solution for the problem, because, as these authors show, for any given value of variance, there is a trade-off between maximizing expected return or skewness. To overcome this difficulty, Briec *et al.* (2016) propose portfolios ranking according to their ability to maximize a three-dimension distance to a benchmark portfolio. Expected return and positive skewness contribute positively to the three-dimension distance, while variance contributes negatively. However, as the authors, themselves, report, this method does not guarantee that an optimum solution for the utility function is obtained.

2.3 Calculation of performance comparison ratios between mean-variance efficient portfolios and portfolios with high skewness

As referred in the introduction, this paper calculates performance comparison ratios between mean-variance efficient portfolios and portfolios with high skewness. The pairwise comparison is based on the Israelson (2004) ratio, which innovates the Sharpe (1966, 1994) ratio. While the Sharpe Ratio measures portfolio performance using the risk-free assets as benchmark, Israelson (2004) extends the benchmarks to risky assets. The innovation of the Israelson (2004) Information Ratio proposed in this paper consists of calculating performance ratios dependent on financial market conditions, which are not taken into consideration in the Israelson Information Ratio. Based on the stock market return, we calculate an upside market Information Ratio, corresponding to the market index return sub-sample above the mean, and a downside market Information Ratio, corresponding to the market index return sub-sample below the mean. The advantage of information ratios dependent on financial market conditions is that they indicate which portfolios offer better protection against loss risk in downside market conditions, and which are more able to benefit from expansion periods in financial markets. To calculate the information ratio we take the difference between the return of an efficient portfolio, $R_{EP,t}$,and the return of a portfolio with high skewness, $R_{HS,t}$, where t is the period in which these values were observed:

$$D_t = \left(R_{EF,t} - R_{HS,t}\right) \quad (14)$$

The mean and standard-deviation of the return difference, defined above, are, respectively:

$$\overline{D} = \sum_{t=1}^{T} \frac{D_t}{T} \quad (15)$$

and

$$\hat{\sigma} = \sqrt{\sum_{t=1}^{T} \frac{\left(D_t - \bar{D}\right)^2}{T-1}} \quad (16)$$

where T is the number of observations. The Information Ratio, *IR* is defined as follows

$$IR = \frac{\bar{D}}{\hat{\sigma}} \quad (17).$$

If the return differences D_t are identically and independently distributed, the difference between the sample mean and expected value, $\bar{D} - E(D)$, follows asymptotically a normal distribution with zero mean and standard deviation $\hat{\sigma}/\sqrt{T}$. Hence, the Information ratio can be subject to the *t* Student's statistics test:

$$\sqrt{T}\left(IR\right) = \sqrt{T}\left(\frac{\bar{D}}{\hat{\sigma}}\right) \quad (18)$$

Taking into consideration the significance level of the test statistics, if the Information Ratio is positive and significantly different from zero, the efficient portfolio performs better than the portfolio with high skewness. Otherwise, if the Information Ratio is negative and significantly different from zero, the portfolio with high skewness performs better than the efficient portfolio. Finally, the result is inconclusive when the Sharpe ratio is not significantly different from zero.

2.3 Comparison between mean-variance efficient portfolios and portfolios with high skewness by probit models

The trade-off between mean-variance optimization and skewness maximization makes them alternate instead of complementary objectives. In these circumstances it is reasonable to choose the portfolio with high probability of offering a higher return than the other. This paper estimates *probit* models to extract the probability that the return of a portfolio with high skewness, $R_{HS,t}$ exceeds the return of a mean-variance efficient portfolio, $R_{EF,t}$. To conduct the estimations, we create binary variables, y_t , which take two alternative values, $y_t = 1$ if $R_{HS,t} > R_{EF,t}$ and $y_t = 0$ otherwise. The estimations are conducted using a latent variable y_t^*, dependent on exogenous variables, X_t. The relation between y_t and the latent variable is defined as follows $y_t = 1$ if $y_t^* > 0$ and $y_t = 0$ otherwise, thus implying that $Prob(y_t = 1) = Prob(y_t^* = 1)$. The relation between the latent variable and the exogenous variables is a linear relation, $y_t^* = X_t\beta + \varepsilon_t$, where ε_t is the residual term, which by assumption follows a normal distribution with zero mean and constant standard deviation σ, *i.e.*, $\varepsilon_t \sim N(0,\sigma)$. According to relation between y_t and y_t^* described above, $Prob(y_t = 1) = Prob(X_t\beta + \varepsilon_t > 1)$. The symmetry of the normal distribution and the statistical properties of ε_t, imply that $Prob(X_t\beta + \varepsilon_t > 1)$ is given by:

$$Prob\left(\frac{\varepsilon_t}{\sigma} < \frac{X_t\beta}{\sigma}\right) = \Phi\left(\frac{X_t\beta}{\sigma}\right) \quad (19)$$

where

$$\Phi\left(\frac{X_t\beta}{\sigma}\right) = \int_{-\infty}^{X_t\beta} \frac{1}{\sqrt{2\pi}} exp\left(\frac{-z^2}{2}\right) dz \quad (20)$$

i.e., $\Phi\left(\frac{X_t\beta}{\sigma}\right)$ is the cumulative value of the standard normal distribution $z \sim N(0,1)$ at the limit $X_t\beta$. The likelihood function, L, to estimate β is defined as follows:

$$L = \prod_{t=1}^{m} \Phi\left(\frac{X_t\beta}{\sigma}\right)^{y_t} \left[1 - \Phi\left(\frac{X_t\beta}{\sigma}\right)^{1-y_t}\right] \quad (21)$$

and the estimations are conducted through the maximization of the logarithm of L:

$$\ln(L) = \sum_{t=1}^{m} \left\{ y_t \Phi\left(\frac{X_t\beta}{\sigma}\right) + (1 - y_t)\left[1 - \Phi\left(\frac{X_t\beta}{\sigma}\right)\right]\right\} \quad (22)$$

Once the parameter β is estimated, $Prob(y_t = 1)$ for each value X_t is given by the standard normal distribution.

The *probit* models in this paper seek to estimate the probability of a portfolio with high skewness offering higher return than a mean-variance efficient portfolio. Several preliminary estimations showed that the best fit was obtained using, as explanatory variable, the third moment of the market index return, scaled through its division by σ_M^3

, where σ_M is the standard-deviation of the return of the stock market index. Hence, the explanatory variable used to estimate the *probit* models is $X_t = \left(R_{M,t} - \bar{R}_M\right)^3 / \sigma_M^3$, which is strictly related to market index skewness.

3 Empirical results

3.1 Data presentation and descriptive statistics

The database used in this paper comprises monthly data of the return on the capital equity shares of forty-four German firms, from January 1999 to December 2019, giving a return series composed of 252 observations per firm. The monthly return series of the German stock price index DAX 40, covering the same period, is the variable used to identify the upside and downside market in the calculation of the Information Ratio of performance comparison between alternate efficient portfolios, and to construct the explanatory variable used in the estimation of *probit* models. The descriptive statistics of the return series of individual firms, comprising mean, standard-deviation, skewness, kurtosis, Jarque-Bera test statistics, and the corresponding significance levels are shown in Table 1.

Insert Table 1 about here

3.2 Mean-variance portfolio selection and the inverse relation between expected return and skewness in efficient portfolios

The determination of mean-variance efficient portfolios, by applying the procedure described in section 2.2, provides the efficient frontier plotted in Figure 2, which shows the optimum combinations of expected return and variance that can be reached in the market under study in this article. Figure 3 plots the relation between skewness and expected return. To plot this curve, we used data of fifty efficient portfolios covering both the upside and the downside segments of the efficient frontier. The correlation between expected return and skewness in this group of efficient portfolios is negative: – 0.822. Figure 3, where expected return is represented in the horizontal axis and skewness in the vertical axis, illustrates that there is an inverse relation between them, except for a small segment, thus confirming the trade-off between expected return and skewness reported in previous literature.

Insert Figure 2 about here

Insert Figure 3 about here

3.3 Performance comparison between efficient portfolios and portfolios with positive skewness and positive expected return

Figure 3 also shows that, in the efficient frontier given by the database used in this article, it is not possible to combine positive expected return with positive skewness. Consequently, to compare the performance of mean-variance efficient portfolios with other portfolios that have both positive skewness and positive expected return, we selected the latter outside the efficient frontier.

We used four efficient portfolios in this comparison, all on the upside segment of the efficient frontier, *i.e.*, with positive expected return. Variance is the criterion of choice of these portfolios, with the purpose of being representative of the entire spectrum of variance that can be found on the entire efficient frontier. The values of variance of this group of efficient portfolios, hereinafter designated respectively as EP_1, EP_2, EP_3 and EP_4 are: $\sigma^2_{EP1} = 0,200\%$, $\sigma^2_{EP2} = 0,400\%$, $\sigma^2_{EP3} = 0,600\%$ and $\sigma^2_{EP4} = 0,800\%$. The descriptive statistics of this group of efficient portfolios, represented in Table 2, show that their skewness, kurtosis, and Jarque-Bera statistics are not significantly different from zero, taking into consideration that their significance levels are above 5%. Hence, these statistics suggest that the return of the efficient portfolios follows the normal distribution.

Insert Table 2 about here

The five assets with the highest skewness, shown in Table 1, were chosen for performance comparison with the four efficient portfolios described above. Those individual assets, hereinafter called HS assets, are identified in column (1) of Table 1, respectively as A1, A19, A35, A36 and A41. The results of the comparative performance information ratio between efficient portfolios and high skewness assets, described in sub-section 2.2, are represented in Table 3, which shows the Information Ratio, and the corresponding significance level for the entire return spectrum and for the upside and downside market described above.

Insert Table 3 about here

IR is positive in all the calculations covering the entire return spectrum, and fifteen out of twenty ratios calculated are significantly different from zero, using the significance level of 5% as the critical level. In downside market, all information ratios are positive and significantly different from zero, meaning that, in all the pairwise comparisons, efficient portfolios perform better than HS assets. Conversely, in upside market conditions, eleven out of twenty ratios are negative, but only three of them are significantly different from zero, and nine ratios are positive, but only four of them are significantly different from zero. According to these results, and given the explanations about IR, presented in sub-section 2.2., all efficient portfolios perform better than HS assets in downside market and, in most of the cases, when the entire return spectrum is considered. In upside market conditions it is not clear that one of the two types of portfolios performs better than the other. The other particularity of the results obtained in the upside market is to include the small number of cases in which HS portfolios perform better than efficient portfolios.

3.4 Results of the comparison between efficient portfolios and HS assets with probit models

In the estimation of *probit* models, we intend to assay the likelihood of the return of an HS asset exceeding the highest return of the four efficient portfolios used in the comparison, hereinafter called Prob(R_{HS}>max(R_{EP})) . This procedure takes into consideration that, in our sample, the return of each HS asset exceeds, on average, the return of four efficient portfolios by 41% of the total observations, while the cases in which its return was between the best and the worst return of efficient portfolios amount to only about 3% of total observations. The binary variable used in the estimations takes the value $y_t = 1$ if $R_{HS,t} > \max\left(R_{EF,t}\right)$ and, $y_t = 0$ otherwise, $\max\left(R_{EF,t}\right)$ being the maximum return of efficient portfolios observed at period t.

As explained above, after some preliminary estimations, the explanatory variable chosen for *probit* final estimations is $X_t = \left(R_{M,t} - \bar{R}_M\right)^3 / \sigma_M^3$ .The results of the *probit* model estimations, represented in Table 4, show that only HS assets A1, A35, and A36 have all the coefficient estimators significantly different from zero. Therefore, only these assets are used for estimating the Prob(R_{HS}>max(R_{EP})).

Insert Table 4 about here

Insert Figure 4 about here

Figures 4.a, 4.b, and 4.c plot the relation between Prob(R_{HS}>max(R_{EP})) and the explanatory variable $X_t = \left(R_{M,t} - \bar{R}_M\right)^3 / \sigma_M^3$ (called in the Figures, for the sake of simplification, Market Index Return 3rd Moment). Those figures illustrate that Prob(R_{HS}>max(R_{EP})) varies positively with $\left(R_{M,t} - \bar{R}_M\right)^3 / \sigma_M^3$. Most of the values of the explanatory variable fall in the segment [-5,+5]. Consequently, Prob(R_{HS}>max(R_{EP})) values associated with this segment of values of the explanatory variable are the most representative of the probability of an HS asset having higher return than efficient portfolios. Prob(R_{HS}>max(R_{EP})) corresponding to the left and right limits of the range [-5,+5] referred above are 6.3% and 70.45%, respectively, in HS A1, 16% and 50%, respectively, in HS A35, and 23,5% and 46,11%, respectively, in HS A36. Consequently, the most frequent values of Prob(R_{HS}>max(R_{EP})) are less than 50%, and values above occur hardly ever.

Conclusions

The Markowitz mean-variance model represented the most important reference of literature on portfolio selection in the last almost 70 years. Deviations of asset returns from normal distribution led several authors to discuss the inclusion of complementary objectives in the mean-variance model. Positive skewness is one of those objectives, because it may be relevant to cautious investors. However, as previous literature reports, there is a trade-off between expected return and skewness that generates incompatibility between the two objectives in optimization procedures based on the mean-variance model.

The results of the empirical analysis conducted in this paper confirm that expected return and skewness vary inversely in mean-variance efficient portfolios. Moreover, efficient portfolios with positive expected return have negative skewness. The tests also show that one cannot reject the hypothesis that the return of efficient portfolios follows the normal distribution. This result suggests that diversification in efficient portfolios significantly reduces the impact of high skewness that may be present in individual assets.

The performance comparison between efficient portfolios and portfolios with positive skewness, shows that the first perform better in general market conditions and offer better protection in negative conditions of the financial market. Only few cases of better performance of portfolios with high skewness were found, when the financial market is booming.

Finally, it is estimated that the likelihood of the return of portfolios with high skewness exceeding the return of efficient portfolios is most frequently below 50%, and values above this hardly ever occur. This result confirms that mean-variance optimization is preferable to positive skewness seeking as method for selecting portfolios.

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Annex A: Tables

Table 1. Descriptive statistics of individual security returns

		Mean	St. Dev	T Stat	Sig .(T Stat.)	Skewness	Sig.(Skew.)	Kurtosis	Sig.(Kurt.)	Jarque-Bera	Sig .(J. Bera)
A1	SAP	0.01148	0.11459	1.59001	0.11309	1.80453	0.00000	12.80308	0.00000	1857.91385	0.00000
A2	SIEMENS	0.00913	0.09407	1.54048	0.12470	0.19211	0.21586	1.74327	0.00000	33.45933	0.00000
A3	DAIMLER	0.00235	0.09463	0.39502	0.69316	0.16383	0.29124	1.14748	0.00025	14.95265	0.00057
A4	ALLIANZ	0.00366	0.09484	0.61299	0.54044	-0.04861	0.75418	4.18775	0.00000	184.24041	0.00000
A5	VOLKSWAGEN	0.00991	0.10820	1.45466	0.14701	-0.36038	0.02025	3.19507	0.00000	112.64345	0.00000
A6	DEUTSCHE TELEKOM	0.00097	0.08472	0.18212	0.85563	0.49964	0.00129	4.45319	0.00000	218.70939	0.00000
A7	ADIDAS	0.01330	0.08000	2.63864	0.00884	-0.21456	0.16691	0.38889	0.21398	3.52145	0.17192
A8	BASF	0.00846	0.07477	1.79616	0.07367	-0.10353	0.50481	0.62754	0.04493	4.58517	0.10101
A9	BAYER	0.00663	0.08258	1.27485	0.20354	-0.18249	0.23975	1.12085	0.00034	14.58981	0.00068
A10	BMW	0.00776	0.08435	1.46138	0.14516	-0.00270	0.98611	0.50991	0.10322	2.73040	0.25533
A11	DEUTSCHE BANK	-0.00083	0.10849	-0.12152	0.90338	0.34547	0.02604	3.43382	0.00000	128.81928	0.00000
A12	MUENCHENER RUCK.	0.00459	0.08278	0.88006	0.37967	0.76352	0.00000	12.40238	0.00000	1639.58468	0.00000
A13	E ON N	0.00111	0.07508	0.23555	0.81398	-0.33742	0.02973	1.39247	0.00001	25.14118	0.00000
A14	RWE	0.00177	0.08746	0.32197	0.74775	-0.11191	0.47095	1.36598	0.00001	20.11802	0.00004
A15	BEIERSDORF	0.00853	0.06029	2.24601	0.02557	0.16030	0.30176	0.35926	0.25096	2.43446	0.29605
A16	CONTINENTAL	0.01235	0.10550	1.85849	0.06427	-0.30145	0.05214	5.65874	0.00000	340.04037	0.00000
A17	MERCK KGAA	0.00976	0.07661	2.02143	0.04430	-0.03409	0.82615	0.79931	0.01064	6.75727	0.03409
A18	HENKEL	0.00684	0.05738	1.89129	0.05974	-0.32795	0.03463	0.55288	0.07728	7.72678	0.02100
A19	ENBW ENGE.BADEN-WURTG.	0.00283	0.05622	0.79788	0.42569	1.88580	0.00000	7.91408	0.00000	807.00565	0.00000
A20	FRESENIUS MED.CARE	0.00810	0.08350	1.53928	0.12500	0.73968	0.00000	6.29591	0.00000	439.18286	0.00000
A21	HANNOVER RUECK	0.00991	0.07624	2.06441	0.04001	-0.18499	0.23336	3.43445	0.00000	125.28918	0.00000
A22	HEIDELBERGCEMENT	0.00614	0.09838	0.99087	0.32270	-0.03379	0.82766	1.83885	0.00000	35.55251	0.00000
A23	FRESENIUS	0.01076	0.08806	1.93990	0.05351	0.75011	0.00000	7.91799	0.00000	681.92495	0.00000
A24	COMMERZBANK	-0.00435	0.13186	-0.52366	0.60097	0.19906	0.19972	2.62426	0.00000	73.97501	0.00000
A25	DEUTSCHE LUFTHANSA	0.00404	0.09366	0.68495	0.49401	-0.28719	0.06429	0.35900	0.25131	4.81735	0.08993
A26	PUMA	0.02066	0.09400	3.48880	0.00007	0.60165	0.00011	1.96015	0.00000	55.54616	0.00000
A27	SARTORIUS	0.02390	0.11533	3.28951	0.00115	0.61241	0.00008	3.09083	0.00000	116.06064	0.00000
A28	BOSS (HUGO)	0.01022	0.10274	1.57874	0.11566	-0.26155	0.09200	1.91087	0.00000	41.21314	0.00000
A29	CECONOMY	-0.00069	0.09802	-0.11108	0.91164	0.23114	0.13648	1.08429	0.00053	14.58852	0.00068

		Mean	St. Dev	T Stat	Sig .(T Stat.)	Skewness	Sig.(Skew.)	Kurtosis	Sig.(Kurt.)	Jarque-Bera	Sig .(J. Bera)
A30	FIELMANN	0.00973	0.06261	2.46725	0.01428	-0.18815	0.22547	0.16117	0.60654	1.75961	0.41486
A31	FUCHS PETROLUB	0.01672	0.08594	3.08903	0.00223	0.17346	0.26380	2.93747	0.00000	91.86562	0.00000
A32	PORSCHE AML.HLDG.PREF.	0.01159	0.11161	1.64773	0.10066	0.71853	0.00000	6.01380	0.00000	401.42503	0.00000
A33	RHEINMETALL	0.01219	0.11197	1.72887	0.08506	0.57493	0.00021	2.84276	0.00000	98.73629	0.00000
A34	THYSSENKRUPP	0.00476	0.10705	0.70525	0.48131	-0.04891	0.75269	0.61526	0.04929	4.07518	0.13034
A35	TUI	0.00310	0.12810	0.38367	0.70155	1.90534	0.00000	15.61306	0.00000	2712.03426	0.00000
A36	1&1 DRILLISCH	0.01879	0.18227	1.63653	0.10298	1.18713	0.00000	4.39859	0.00000	262.33977	0.00000
A37	ADIDAS (XET)	0.01326	0.08092	2.60200	0.00982	-0.24204	0.11893	0.37971	0.22499	3.97445	0.13707
A38	AIXTRON	0.01281	0.17016	1.19469	0.23334	0.20821	0.17983	0.32539	0.29843	2.93244	0.23080
A39	AURUBIS	0.00978	0.08194	1.89490	0.05925	-0.16981	0.27399	0.43494	0.16458	3.19733	0.20217
A40	AXEL SPRINGER	0.00674	0.07690	1.39120	0.16540	0.33407	0.03139	1.33526	0.00002	23.40795	0.00001
A41	COMPUGROUP MEDICAL N	0.02494	0.12366	3.20121	0.00155	1.34930	0.00000	5.04135	0.00000	343.32544	0.00000
A42	DAIMLER (XET)	0.00237	0.09459	0.39799	0.69097	0.20484	0.18697	1.34637	0.00002	20.79570	0.00003
A43	DMG MORI	0.01482	0.11101	2.11998	0.03499	0.11789	0.44758	1.79472	0.00000	34.40448	0.00000
A44	DUERR	0.01193	0.10623	1.78311	0.07578	0.27631	0.07507	0.93823	0.00272	12.44934	0.00198

Table 2. Descriptive statistics of mean-variance efficient portfolios

	Mean	St. Dev	Variance	T .Stat	Sig .(T Stat.)	Skewness	Sig.(Skew.)	Kurtosis	Sig.(Kurt.)	Jarque-Bera	Sig .(J. Bera)
EP1	0.02525	0.02525	0.00201	8.94567	0.00000	-0.18708	0.22814	0.44247	0.15739	3.52556	0.17157
EP2	0.03643	0.06337	0.00402	9.12451	0.00000	-0.21108	0.17388	0.27672	0.37655	2.67539	0.26245
EP3	0.04469	0.07761	0.00602	9.13964	0.00000	-0.22174	0.15316	0.18705	0.55004	2.43237	0.29636
EP4	0.05155	0.089621	0.00803	9.13148	0.00000	-0.22762	0.14255	0.13086	0.67583	2.35583	0.30792

Table 3: Comparative Performance Information Ratio (IR) between efficient portfolios and high skewness portfolios

Eff. Portfolio	HS Asset	Financial Market Conditions					
		Entire return spectrum		Upside market		Downside market	
		IR Value	Sig. Level	IR Value	Sig. Level	IR Value	Sig. Level
EP1	A1	0.11728	0.06380	-0.25566	0.00482	0.65364	0.00000
	A19	0.30742	0.00000	0.32541	0.00038	0.29062	0.00143
	A35	0.16279	0.01033	-0.17359	0.05359	0.56194	0.00000
	A36	0.03578	0.57060	-0.31106	0.00066	0.52267	0.00000
	A41	0.00269	0.96598	-0.16321	0.06932	0.20392	0.02376
EP2	A1	0.20517	0.00128	-0.11368	0.20432	0.64943	0.00000
	A19	0.40780	0.00000	0.49643	0.00000	0.32537	0.00038
	A35	0.23582	0.00023	-0.04615	0.60536	0.56619	0.00000
	A36	0.09677	0.12575	-0.21961	0.01506	0.53999	0.00000
	A41	0.09751	0.12288	-0.02779	0.75558	0.24619	0.00658
EP3	A1	0.26219	0.00004	-0.01567	0.86070	0.63435	0.00000
	A19	0.45683	0.00000	0.58617	0.00000	0.33781	0.00023
	A35	0.28316	0.00001	0.04056	0.64966	0.56216	0.00000
	A36	0.14009	0.02705	-0.15293	0.08853	0.54579	0.00000
	A41	0.16350	0.01000	0.07091	0.42759	0.26976	0.00299
EP4	A1	0.30358	0.00000	0.05978	0.50343	0.61798	0.00000
	A19	0.48552	0.00000	0.64288	0.00000	0.34250	0.00019
	A35	0.31787	0.00000	0.10699	0.23203	0.55573	0.00000
	A36	0.17436	0.00606	-0.09878	0.26962	0.54688	0.00000
	A41	0.21366	0.00081	0.14955	0.09570	0.28455	0.00178

Table 4: *Probit* model estimations of the probability that HS portfolios return exceeds efficient portfolios return

HS portfolio	Explanatory variable	Coefficient (Stat. error in brackets)	Likelihood Ratio test of coefficients (Sig. level in brackets)	Log Likelihood of estimation
A1	Constant	-0.4829 *** (0.0864)	29.4481 (0.0000)	-143.5173
	$\frac{\left(R_{M,t}-\bar{R}_M\right)^3}{\sigma_M^3}$	0.20993*** (0.05675)		
A19	Constant	-0.60560 *** (0.0849)	0.3119 (0.5765)	-147.771
	$\frac{\left(R_{M,t}-\bar{R}_M\right)^3}{\sigma_M^3}$	-0.00880 (0.01587)		
A35	Constant	-0.5012 *** (0.0848)	13.9427 (0.0002)	-148.9459
	$\frac{\left(R_{M,t}-\bar{R}_M\right)^3}{\sigma_M^3}$	0.09871*** (0.03541)		
A36	Constant	-0.4102 *** (0.0827)	10.1610 (0.0014)	-156.0053
	$\frac{\left(R_{M,t}-\bar{R}_M\right)^3}{\sigma_M^3}$	0.06255*** (0.02260)		
A41	Constant	-0.3491 *** (0.0810)	1.1657 (0.2803)	-164.2390
	$\frac{\left(R_{M,t}-\bar{R}_M\right)^3}{\sigma_M^3}$	0.01739 (0.01662)		

Annex B: Figures

Figure 1: Efficient frontier

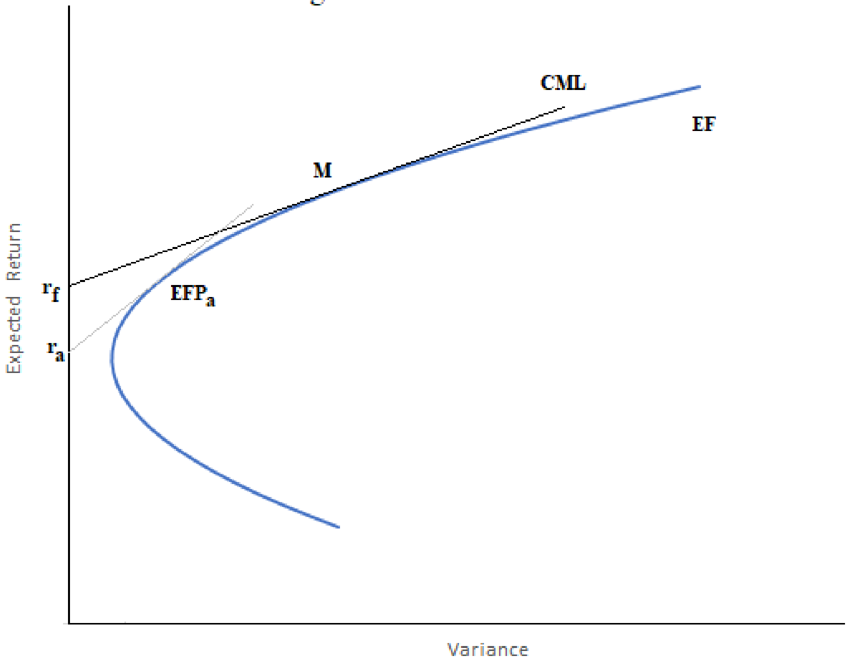


Figure 2: Efficient frontier of the German stock market

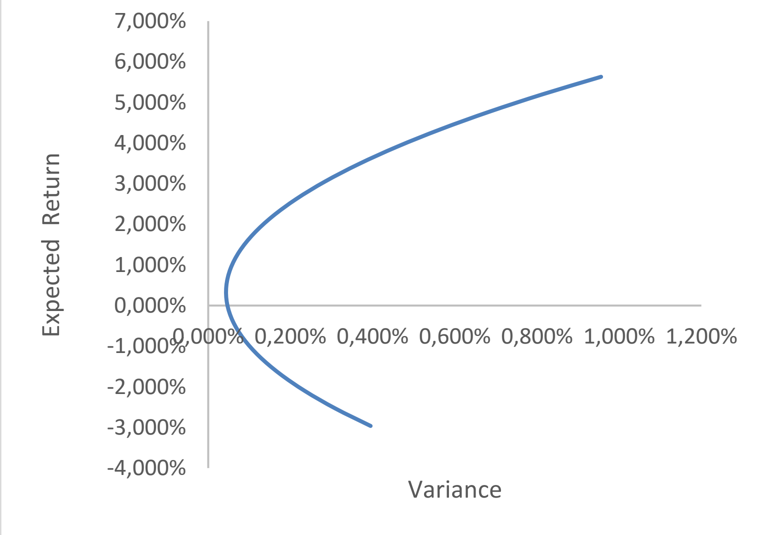
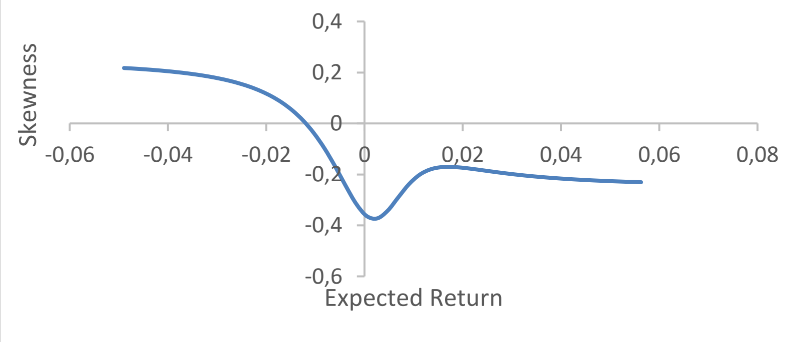


Figure 3: Expected Return and Skewness of Efficient Portfolios



Figures 4.a, 4.b and 4.c: Probability of HS asset return to exceed efficient portfolio return

Figure 4.1 Probability of HS asset A1 return to exceed efficient portfolios return

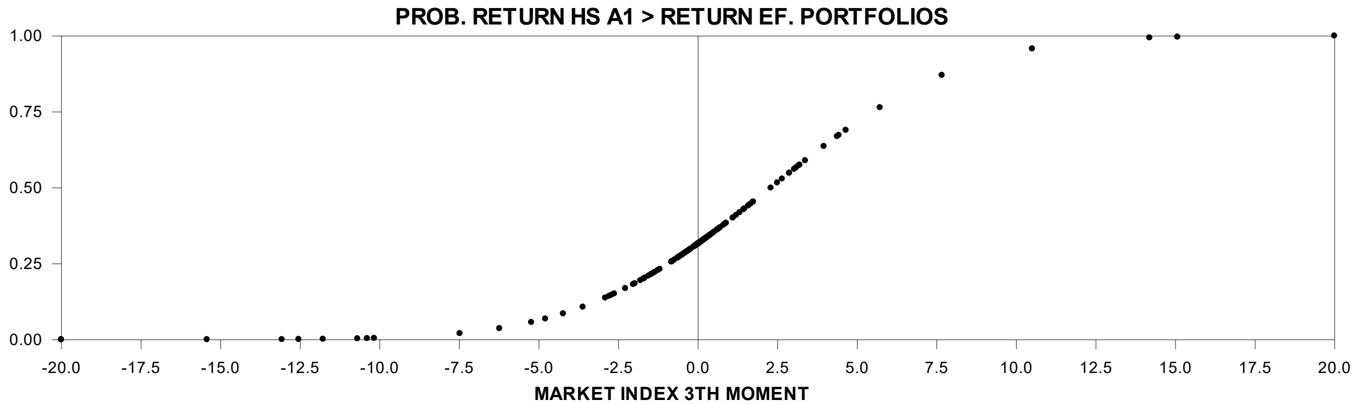


Figure 4.2 Probability of HS asset A35 return to exceed efficient portfolio return

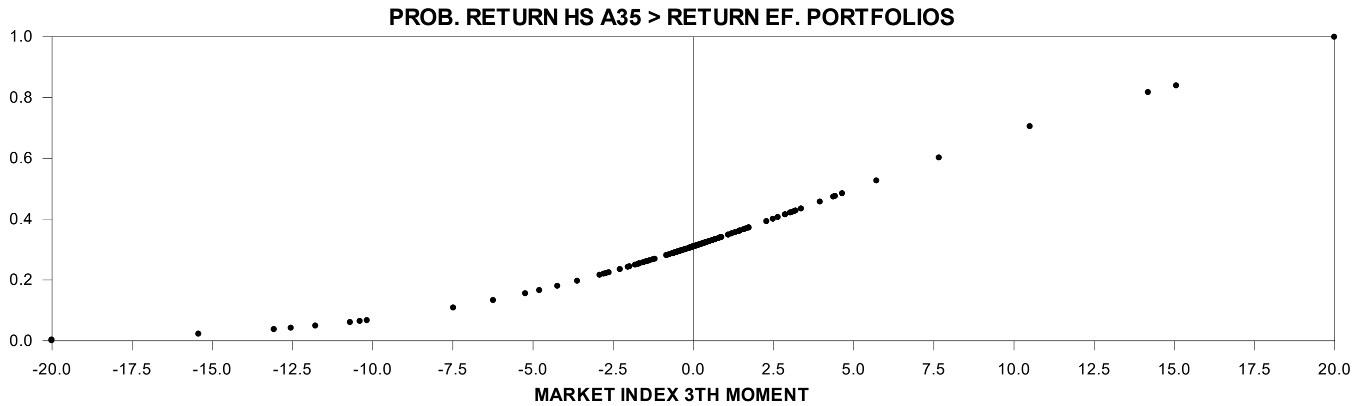


Figure 4.3 Probability of HS asset A36 return to exceed efficient portfolio return